Instantaneous Companding of Quantized Signals

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(Manuscript received October 8, 1956)

Instantaneous companding may be used to improve the quantized approximation of a signal by producing effectively nonuniform quantization. A revision, extension, and reinterpretation of the analysis of Panter and Dite permits the calculation of the quantizing error power as a function of the degree of companding, the number of quantizing steps, the signal volume, the size of the "equivalent dc component" in the signal input to the compressor, and the statistical distribution of amplitudes in the signal. It appears, from Bennett's spectral analysis, that the total quantizing error power so calculated may properly be studied without attention to the detailed composition of the error spectrum, provided the signal is complex (such as speech or noise) and is sampled at the minimum information-theoretic rate.

These calculations lead to the formulation of an effective process for choosing the proper combination of the number of digits per code group and companding characteristic for quantized speech communication systems. An illustrative application is made to the planning of a hypothetical PCM system, employing a common channel compandor on a time division multiplex basis. This reveals that the calculated companding improvement, for the weakest signals to be encountered in such a system, is equivalent to the addition of about 4 to 6 digits per code group, i.e., to an increase in the number of uniform quantizing steps by a factor between $2^4 = 16$ and $2^6 = 64$.

Comparison with the results of related theoretical and experimental studies is also provided.

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(Passages marked with an asterisk contain mathematical details which may be omitted in a first reading without loss of continuity.)

I. INTRODUCTION

Quantized pulse modulation has been the subject of considerable attention in the last decade.[1-17] Proposals for practical application of such modulation usually provide for the transmission, by time division multiplex, of a class of signals covering an extensive power range.[1, 6] Such proposals almost invariably assign a vital role to instantaneous compandors. The present discussion is devoted to the formulation of general quantitative criteria for the choice of a suitable companding characteristic.

A. *Fundamental Properties of Pulse Modulation**

1. Unquantized Signals

Unquantized pulse signals are produced when a band-limited signal (such as low-pass filtered speech) is sampled instantaneously at a rate greater than or equal to the minimum acceptable value of slightly more than twice the top signal frequency. The transmission of the continuous range of pulse amplitudes so produced is known as pulse amplitude modulation (PAM). Alternatively one may translate the sampled amplitudes into variations either in the width of periodic pulses of constant amplitude (pulse duration modulation or PDM), or in the spacing of pulses of uniform amplitude and width (pulse position modulation or PPM). Regardless of the mode of transmission, the unquantized signal pulses are sensitive to noise in the transmission medium.

2. Quantized Signals (PCM)

Although sampling constitutes temporal quantization, it is convenient to adhere to conventional usage (as codified by Bennett[2] and Black[1]) in restricting the designation "quantized signals" to those which have been quantized in amplitude, as well as sampled in time, in order to permit encoded (i.e., essentially telegraphic) transmission. Thus a finite range of possible signal amplitudes, large enough to accommodate the strongest signal to be encountered in a given application, may be divided into N equal parts or quantizing steps. Each instantaneous pulse amplitude of a PAM signal is then compared with this ladder-like array; amplitude quantization is accomplished by replacing all amplitudes falling in any portion of a quantizing step by a single value uniquely characterizing that interval.

* This brief account is intended merely to specify the minimum amount of background information required to avoid confusion in the present discussion. Details may be found in the many excellent and readily accessible references already cited.

Use of a binary number representation permits the encoded transmission of the N possible quantized amplitudes in terms of groups of on-off pulses containing n binary digits per code group (where $N = 2^n$).* These pulses may be considered impervious to noise in the transmission medium in the sense that complete information is conveyed by the mere recognition of the presence or absence of a pulse rather than a determination of a precise magnitude. Consequently such pulses may, in principle, repeatedly be regenerated in the transmission medium, provided that regeneration occurs before the on-off pulses have been rendered indistinguishable from each other by noise.

The designation pulse code modulation (PCM) may therefore be used synonymously with quantized pulse modulation to distinguish the latter from the previously defined varieties of unquantized pulse modulation. In view of the restriction of present interest to the role of quantization *per se*, there is no need to proceed beyond the choice of quantized PAM as the prototype PCM signal in this discussion, in spite of the fact that PDM and PPM may also be quantized to yield PCM.[1]

B. Quantizing Impairment in PCM Systems

From the foregoing it is clear that quantization (i.e., the representation of a bounded continuum of values by a finite number of discrete magnitudes), permits the encoded, and therefore essentially noise-free transmission of approximate, rather than exact values of sampled amplitudes. In fact, *the deliberate error imparted to the signal by quantization is the significant source of PCM signal impairment.*[1-5] Adequate limitation of this quantization error is therefore of prime importance in the application of PCM to communication systems.

A number of methods of reducing quantizing error suggest themselves on a purely qualitative and intuitive basis. For example, one may obtain a finer-grained approximation by providing more, and therefore smaller, quantizing steps for a given range of amplitudes. Alternatively, one may provide a more complete description of the signal by increasing the sampling rate beyond the minimum information-theoretic value already assumed.†

It is also possible to vary the size of the quantizing steps (without adding to their number) so as to provide smaller steps for weaker signals.

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* Of course, number representations using a base, b, other than two, so that $N = b^n$, are also available. These are presently of academic interest in view of the increased complexity of instrumentation they imply.[3]

† See Fig. 5 of Reference 2 for a quantitative evaluation of the efficacy of this measure.

Whereas the first two techniques result in an increase of bandwidth and system complexity, the third requires only a modest increase in instrumentation without any increase in bandwidth.* This investigation is therefore devoted to the study of nonuniform step size as a means of reducing quantizing impairment.

C. *Physical Implications of Nonuniform Quantization*

1. *Quantizing Error as a Function of Step Size*

Quantizing impairment may profitably be expressed in terms of the total mean square error voltage since the ratio of the mean square signal voltage to this quantity is equal to the signal-to-quantizing error power ratio.

In evaluating the mean square error voltage, we begin by considering a complex signal, such as speech at constant volume, whose pulse samples yield an amplitude distribution corresponding to the appropriate probability density. These pulse samples may be expected to fall within, or "excite", all the steps assigned to the signal's peak-to-peak voltage range. It will be assumed that, for quality telephony, the steps will be sufficiently small, and therefore numerous, to justify the assumption that the probability density is effectively constant within each step, although it may be expected to vary from step to step. Thus the continuous curve representing the probability density as a function of instantaneous amplitude is to be replaced by a suitable histogram.

If the midstep voltage is assigned to all amplitudes falling in a particular quantizing interval, the absolute value of the error in any pulse sample will be limited to values between zero and half the size of the step in question; when combined with the assumed approximation of a uniform probability density within each step, this choice minimizes the mean square error introduced at each level.[5] Summation of the latter quantity over all levels then yields the result that the total mean square quantizing error voltage is equal to one-twelfth the weighted average of the square of the size of the voltage steps traversed (i.e., excited) by the input signal. The direct consideration of the physical meaning of this result (which, as (6) below, will constitute the basis of all subsequent calculations) will now be shown to provide a simple qualitative description of the implications of nonuniform quantization.

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* We refer to bandwidth in the transmission medium as determined by the pulse repetition rate, which, in the time division multiplex applications envisioned herein, is given by the product: (sampling rate) × (number of digits or pulses per sample) × (number of channels).

2. Properties of the Mean Square Excited Step Size

Fig. 1(a) shows the range of input voltages, between the values $+V$ and $-V$ divided into N equal quantizing steps (i.e., uniformly quantized); Fig. 1(b) depicts the same range divided into N tapered steps, corresponding to nonuniform quantization.

Consider a complex signal, such as speech, whose distribution of instantaneous amplitudes at constant volume results in the excitation (symmetrically about the zero level) of the steps in the moderately large interval $X-X'$. The quantizing error power will be shown to be proportional to the (weighted) average of the square of the excited step size. For uniform quantization, it is clear, from Fig. 1(a), that this average is a constant, independent of the statistical properties of the signal. For a nonuniformly quantized signal, [Fig. 1(b)], the mean square excited step size is reduced by the division of the identical interval $X-X'$ into more steps, most of which are smaller than those shown in Fig. 1(a). Appreciation of the full extent to which the quantizing error power may so be reduced requires the added recognition that the few larger quantizing steps in the range $X-X'$, corresponding to excitation by comparatively rare speech peaks, are far less significant in their contribution to the weighted average than the small steps in the vicinity of the origin, due to the nature of the probability density of speech at constant volume.[18]

It is also clear that weaker signals, corresponding to a contraction of the interval $X-X'$, enjoy the greatest potential tapering advantage since their excitation may be confined to steps which are all appreciably smaller than those in Fig. 1(a). However, if the interval $X-X'$ were to

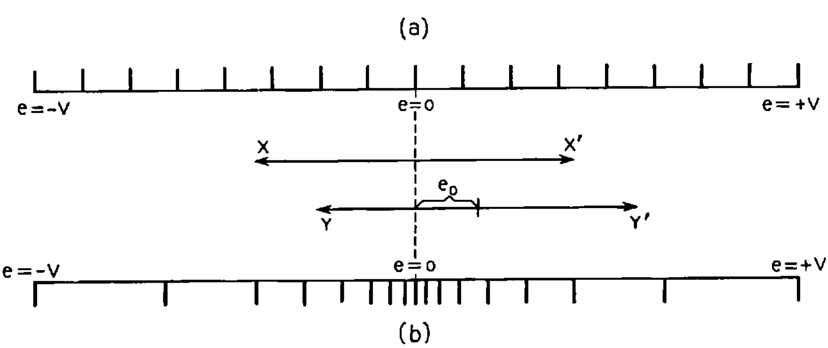

Fig. 1 — (a) Distribution of steps of equal size corresponding to direct, uniform quantization; (b) nonuniform quantization of this range into the same number of steps. The function of the instantaneous compandor is to provide such nonuniform quantization in the manner illustrated in Fig. 2.

increase in size and approach the full range, $+V$ to $-V$, (to accommodate stronger signals), the excitation of extremely large steps might result in an rms step size exceeding the uniform size shown in Fig. 1(a).

Fig. 1 also indicates that signals (including unwanted noise) too weak to excite even the first quantizing step (and therefore absolutely incapable of transmission) when uniformly quantized, may successfully be transmitted as a result of the excitation of a few steps following nonuniform quantization.

Although the assumption that the average value of the signal is zero is quite proper for speech, subsequent discussion will disclose the possibility that the quiescent value of the signal, as it appears at the input to the quantizing equipment, may not always coincide with the exact center of the voltage range depicted in Fig. 1. This effect may formally be described in terms of the addition of an equivalent dc bias to the speech input at the quantizer. As shown in Fig. 1, the addition of such a dc component, e_0, to the signal which previously excited the band of steps labeled $X-X'$, transforms $X-X'$ into an array of equal extent $Y-Y'$, centered about $e = e_0$ instead of $e = 0$. This causes the excitation of some larger steps, in Fig. 1(b), as well as the assignment of greatest weight[18] to the steps in the vicinity of $e = e_0$, which are larger than those near $e = 0$; the net result is an increase in the rms excited step size, and the quantizing error power. This effect will depend on the comparative size of e_0 and the signal as well as on the degree of step size variation. In particular, Fig. 1(a) indicates that the presence of e_0 does not affect the rms excited step size under conditions of uniform quantization.

It is clear from the foregoing that the effect of nonuniform quantization of PCM signals will vary greatly with the strength of the signal; greatest improvement is to be expected for weak signals, whereas an actual impairment may be experienced by strong signals. The range of signal volumes is therefore of prime importance in the choice of the proper distribution of step sizes.

D. *Nonuniform Quantization Through Uniform Quantization of a Compressed Signal*

Nonuniform quantization is logically equivalent to uniform quantization of a "compressed version" of the original input signal. When applied directly, tapered quantization provides an acceptably high ratio of sample amplitude to sample error for weak pulses, by decreasing the errors (i.e., the step sizes) assigned to small amplitudes. Signal compression achieves the same goal by increasing weak pulse amplitudes without altering the step size.

The instantaneous compressor envisioned herein is, in essence, a nonlinear pulse amplifier which modifies the distribution of pulse amplitudes in the input PAM signal by preferential amplification of weak samples. A satisfactory compression characteristic will have the general shape shown in Fig. 2. Thus the amplification factor, (v/e), varies from a large value for small inputs to unity for the largest amplitude (V) to be accommodated, so that the distribution of pulse sizes may be modified without changing the total voltage range. Fig. 2 also illustrates how uniform quantization of the compressor output produces a tapered array of input steps similar to those already considered in connection with Fig. 1(b).

A complementary device, the expandor, employs a characteristic inverse to that of the compressor to restore the proper (quantized) distribution of pulse amplitudes after transmission and decoding. Taken together, the compressor and expandor constitute a compandor.

The resolution of tapered quantization into the sequential application of compression, uniform quantization, and expansion is operationally convenient,[6] as well as logically sound. Since there is a one-to-one correspondence between step size allocations and compression characteristic

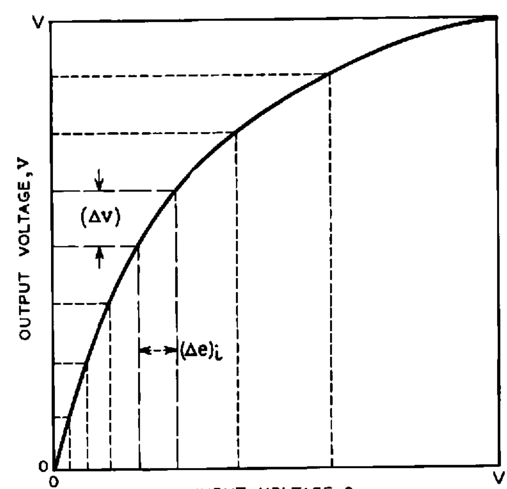

Fig. 2 — Curve illustrating the general shape of a suitable instantaneous compression characteristic. All continuous, single-valued curves connecting the origin to the point (V, V) and rising from the origin with a slope greater than one, i.e., $(dv/de)_{e=0} > 1$, are potential compression characteristics. The symmetrical negative portion $[v(e) = -v(-e)]$ is not shown. The production of a tapered array of input steps $(\Delta e)_i$ by uniform quantization of the output into steps of (equal) size Δv, is also represented.

curves, the central problem of choosing the proper distribution of step sizes will be discussed in terms of the choice of the appropriate compression characteristic; the reduction of quantizing error, corresponding to nonuniform quantization without change in the total number of steps, will be termed companding improvement.

E. *The Mechanism of Companding Improvement in Various Communication Systems*

1. *Syllabic Companding of Continuous Signals*[19, 20]

Originally, the compandor consisted of a compressor and complementary expandor operating at a syllabic rather than instantaneous rate in frequency division systems, since instantaneous companding was found to imply an undesirable increase in bandwidth in such systems.[19]

In spite of the existence of syllabic power variations, a useful understanding of such compandor action may be inferred from the consideration of the long-time average power. Thus, in its simplest form, the compressor might provide amplification varying from a constant value within the range of volumes corresponding to weak speech to little or no amplification for comparatively strong signals prior to transmission. Although it is an amplifying device, the compressor takes its name from the contraction of the transmitted volume range which results from selective amplification of the weakest signals. Since the distortion of the signal by the compressor may virtually be confined to a change in loudness, the compressor output may be expected to be intelligible.

In interpreting a compression characteristic, syllabic application permits the identification of the ordinate and abscissa with $\sqrt{\overline{v^2}}$ and $\sqrt{\overline{e^2}}$, rather than v and e as shown in Fig. 2. This substitution of rms for instantaneous signals not only confines the significance of the compression characteristic to the first quadrant but also removes the need for compandor response to input signals below some small, nonzero, threshold value.

If we designate the mean square noise voltage in the transmission medium by $\overline{v_n{}^2}$, the amplification of weak signals prior to exposure to this noise provides an increase in the transmitted signal to noise ratio from $(\overline{e^2}/\overline{v_n{}^2})$ to $(\overline{v^2}/\overline{v_n{}^2})$, i.e., by a factor of $(\overline{v^2}/\overline{e^2})$. This increase in signal-to-noise ratio may be read directly from the graph of the compression characteristic, and is unaffected by the identical treatment accorded signal and noise at the expandor. Furthermore, noise received during the silent intervals, between speech bursts, is attenuated by the expandor.

Under these circumstances it is appropriate to resolve companding improvement into the separate contributions of an increased signal to noise ratio for weak speech by the compressor, and a quieting of the circuit in the absence of speech by the expandor. The introduction of an independent source of noise in the channel between the compressor and expandor is the key to such behavior.

2. *Instantaneous Companding of Unquantized Pulse Signals*

Time division systems, employing unquantized pulse modulation (e.g., PAM) are admirably suited to the application of instantaneous companding to the individual pulse samples. Since each pulse is amplified to a degree which varies with its input amplitude, the compressor output is a sampled version of a distorted signal.

As in the syllabic case, the location of the noise source in the channel between the compressor and expandor permits an improvement of the received signal-to-noise ratio for weak signals. Furthermore, the expandor again assumes the separate and distinct task of suppressing channel noise in the absence of speech.

Unfortunately, quantitative expression of the companding improvement is not as simple as in the syllabic case. The response to instantaneous amplitudes much lower than the rms threshold signal (including zero) becomes important and the improvement factor may not (except in the special case of a linear compression characteristic) simply be read from a graph relating instantaneous values of v and e. Instead, one must employ the probability density of the signal in order properly to account for the distinctive treatment accorded individual pulse amplitudes in a complex signal.

3. *Instantaneous Companding of Quantized Signals*

Although the same physical devices which serve as an instantaneous compressor and expandor in a PAM system may also be used in a PCM system, the functional description of companding improvement is different in the two applications. Whereas the compandor is used to combat channel noise in a PAM system, encoded transmission permits a PCM system to assign this task to the devices which transmit and regenerate code pulses. Thus, assuming that error-free encoded transmission is realized, the quantized signal may be regarded as completely impervious to noise in the transmission medium. Quantization is required to permit such transmission. The sole purpose of the PCM compandor is to reduce the quantizing impairment of the signal by converting uniform to effectively nonuniform quantization.

Although the expandor continues to collaborate with the compressor in improving the quality of weak signals, it is now neither necessary nor possible for it to perform the separate function of quieting the circuit in the absence of speech. Indeed, apart from instrumentational difficulties which might arise, it is conceptually sound to transfer the PCM expandor to the transmitting terminal, with expansion taking place subsequent to quantization but prior to encoding and transmission. Another interesting peculiarity of the PCM expandor is the restriction of its operation, by quantization, to a finite number of discrete operating points on the continuous characteristic.

The use of companding to reduce the quantizing error which owes its very existence to, and is therefore a function of, the signal, is thus significantly different from the use of companding to reduce the effects of an independent source of noise in the transmission medium.

F. *Applicability of the Present Analysis*

Before we proceed to a detailed analysis, it is important to emphasize certain restrictive conditions required for the meaningful application of the results to be derived.

1. Signal Spectrum

A signal with a sufficiently complex spectrum, such as speech, is required to justify consideration of the total quantizing error power without regard to the detailed composition of the error spectrum. Although it is known that quantization of simple signals (e.g., sinusoids) results in discrete harmonics and modulation products deserving of individual attention,[8, 9] Bennett has shown that the error spectrum for complex signals is sufficiently noise-like to justify analysis on a total power basis.[2, 12]

2. Sampling Rate

The consistent comparison of signal power with the total quantizing error power, rather than with the fraction of the latter quantity appropriate to the signal band, might at first appear to impose serious limitations on the present analysis. Furthermore, the role of sampling has not been discussed explicitly. It is therefore important to note that the justification for this treatment, in the situation of actual interest, has also been given by Bennett.[2] We need only add the standard hypothesis[1-6] that the sampling rate chosen for a practical system would equal the minimum acceptable rate (slightly in excess of twice the top signal frequency[3]) in order to invoke Bennett's results, which tell us that, for this sampling rate, the quantizing error power in the signal band and the total quantizing error power are identical.[2] Thus, sampling at the minimum rate is assumed throughout.

3. Number of Quantizing Steps

As already remarked, the present results are based on the assumption that N is not small, inasmuch as we assume a probability density which, although varying from step to step, remains effectively constant within each quantizing step; indeed the step sizes will be treated as differential quantities.

Experimental evidence[6, 7, 10] (as well as the analysis to follow) argues against the consideration of fewer than five digits (i.e., $2^5 = 32$ quantizing steps) for high quality transmission of speech. Numerical estimates indicate that the present approximation should be reasonable for five or more digits per code group. These estimates are confirmed by the consistency of actual measurements of quantizing error power with calculations based on the same approximation (see Fig. 8 of reference 2 for 5, 6, and 7 digit data obtained with an input signal consisting of thermal noise instead of speech).

Further indication of the adequacy of this approximation is provided by the knowledge that Sheppard's corrections (see Section II-B) appear adequate even when (Δe) is not very small, for a probability density which (as is the case for speech[18]) approaches zero together with its derivatives at both ends of the (voltage) range under consideration.[24]

Therefore, we are not presently concerned with the limitations imposed by this approximation.

4. Subjective Effects Beyond the Scope of the Present Analysis

We shall have occasion to study graphs depicting the signal to quantizing error power ratio as a function of signal power. Although these curves, and the equations they represent, will always be of interest for the case where even the weakest signal greatly exceeds the corresponding error power, there exists the possibility of rash extrapolation to the region where this inequality is reversed. Unfortunately, such extrapolation may have little or no meaning.* This is particularly clear when one considers that signals incapable of exciting at least the first quantizing step, in the absence of companding, will be absolutely incapable

* This is implicit in the deduction of Equation (6).

of transmission. Under these circumstances, companding may actually *resuscitate* a signal; the mathematical description of resuscitation (as anything short of infinite improvement) is clearly beyond the scope of the present analysis.

At the other extreme, it is probable that there exists a limit of error power suppression beyond which listeners will fail to recognize any further improvement. Our analysis will not be useful in describing this region of subjective saturation. Furthermore, it is possible that the subjective improvement afforded a listener by adding to the number of quantizing steps, or companding, may depend on the initial and final states, even before subjective saturation is reached. For example, it is entirely possible that the change from 5 to 6 digits per code group may provide a degree of improvement which appears different to the listener from that corresponding to the increase from 6 to 7 digits, although the present mathematical treatment does not recognize such a distinction.

II. EVALUATION OF MEAN SQUARE QUANTIZATION ERROR (σ)

A.* *Generalization of the Analysis of Panter and Dite*

The mean square error voltage, σ_j, associated with the quantization of voltages assigned to the j^{th} voltage interval, e_j, is adopted as the significant measure of the error introduced by quantization. If e_j is to represent any voltage, e, in the range

$$Q_j = \left[e_j - \frac{(\Delta e)_j}{2}\right] \leqq e \leqq \left[e_j + \frac{(\Delta e)_j}{2}\right] = R_j \tag{1}$$

then

$$\sigma_j = \int_{Q_j}^{R_j} (e - e_j)^2 P(e)\, de \tag{2}$$

where $(e - e_j)$ is the voltage error imparted to the sample amplitude by quantization and $P(e)$ is the probability density of the signal. The location of e_j at the center of the voltage range assigned to this level minimizes σ_j since we shall assume an effectively constant value of $P(e)$ within the confines of a single step.

If the value of $P(e)$ is approximated by the constant value $P(e_j)$ appropriate to e_j in (2), it follows that

$$\sigma_j = (\Delta e)_j^3 P(e_j)/12 \tag{3}$$

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* This passage contains mathematical details which may be omitted, in a first reading, without loss of continuity.

The total mean square voltage error, σ, is equal to the sum of the mean square quantizing errors introduced at each level, so that,

$$\sigma = \sum_j \sigma_j = \tfrac{1}{12}\sum_j P(e_j)(\Delta e)_j^3 \tag{4a}$$

$$= \tfrac{1}{12}\sum_j (\Delta e)_j^2[P(e_j)(\Delta e)_j] \tag{4b}$$

which may be rewritten as

$$\sigma \cong \tfrac{1}{12}\sum_j (\Delta e)_j^2 p_j \tag{4c}$$

since the discrete probability appropriate to the j^{th} step is given by

$$p_j = \int_{Q_j}^{R_j} P(e)\, de \cong [P(e_j)(\Delta e)_j] \tag{5}$$

Hence,

$$\sigma \cong \tfrac{1}{12}[(\Delta e)^2]_{\text{AV}} = \overline{(\Delta e)^2}/12 \tag{6}$$

Thus, the total mean square error voltage is equal to one-twelfth the average of the square of the input voltage step size when the steps are sufficiently small (and therefore numerous) to justify the approximations employed in the deduction of (6). In applying (6), it is important to note that (4) implies that this is a *weighted* average over the steps traversed (or "excited") by the signal.

In the special case of uniform step size, substitution of $(\Delta e)_j = (\Delta e) =$ const. reduces (6) to the simple form

$$\sigma_0 = [\sigma]_{\Delta e = \text{const}} = (\Delta e)^2/12 \tag{7}$$

Equations (6) and (7) provide the basis for the qualitative interpretation of quantizing error power which has already been discussed in connection with Fig. 1.

The deduction of (6) from (4a) is implicit in the work of Panter and Dite.[5] The absence of an explicit formulation of (6) therein* results from the direct application of the equivalent of (4b) to a specific problem involving a particular algebraic expression for $(\Delta e)_j$.

A prior, equivalent derivation of (7), based on a graphical representation of $(e - e_j)$ as a sawtooth error function for uniform quantization has been given by Bennett.[2] Although this derivation bypassed (6), Bennett has also analyzed compressed signals by means of an expression

* The present notation has been chosen to resemble that of Reference 5 in order to facilitate direct comparison by the reader.

[(1.6) of Reference 2] which is equivalent to (6), when the average is expressed as an integral over a continuous probability distribution and (Δe) is replaced by $(de/dv)(\Delta v)$, with (Δv) = const. This form of (6) is the point of departure for the calculation in the Appendix.

B. *Operational Significance of σ*

Manipulation of (2) may be shown to result in the expression

$$\sigma = \sum_j \sigma_j = \sum_j e_j^2 p_j - \int e^2 P(e)\, de,$$

which is the difference between the mean square signal voltages following and preceding quantization. Hence σ is proportional to the difference between the quantized and unquantized signal powers. Since σ is intrinsically positive, the quantizing error power is *added* to the signal by quantization and is, in principle, measurable as the difference between two wattmeter readings.

In addition to providing an operational interpretation of the quantizing error power, the rewritten expression for σ reveals the equivalence of σ to the "Sheppard correction" to the grouped second-moment in statistics,[24-27] where calculations are facilitated by grouping (i.e., uniform quantization) of numerical data. The reader who is interested in a more elaborate deduction of (7) from the Euler-Maclaurin summation formula, as well as discussions of the validity of (7), may therefore consult the statistical literature.

III. CHOICE OF COMPRESSION CHARACTERISTIC

A. *Restriction to Logarithmic Compression*

We shall consider the properties of the logarithmic type of compression characteristic,* defined by the equations†

$$v = \frac{V \log [1 + (\mu e/V)]}{\log (1 + \mu)}, \qquad \text{for} \qquad 0 \leqq e \leqq V \tag{8a}$$

and

$$v = \frac{-V \log [1 - (\mu e/V)]}{\log (1 + \mu)}, \qquad \text{for} \qquad -V \leqq e \leqq 0 \tag{8b}$$

* The author first encountered this characteristic in the work of Panter and Dite[6] and the references thereto cited by C. P. Villars in an unpublished memorandum. He has since learned that such characteristics had been considered by W. R. Bennett as early as 1944 (unpublished), as well as by Holzwarth[16] in 1949.

† Unless otherwise specified, natural logarithms will be used throughout.

In (8), v represents the output voltage corresponding to an input signal voltage e, and μ is a dimensionless parameter which determines the degree of compression.

Typical compression characteristics, corresponding to various choices of the compression parameter, μ, in (8a), are shown in Fig. 3. The logarithmic replot of Fig. 4 provides an expanded picture of small amplitude behavior, as well as evidence of the probable realizability of such characteristics.

Although restriction of attention to (8) may at first appear to impose serious limitations on the generality of the analysis, this impression is not confirmed by more careful scrutiny of the problem.

Perusal of Fig. 3 indicates that (8) generates a considerable variety of curves which meet the general requirements already enunciated in connection with Fig. 2. Thus, the constant factor, $V/\log(1 + \mu)$, has been chosen to satisfy the condition

$$[v]_{e=V} = V \tag{9}$$

Evidence of the significance of the μ-characteristics may be derived

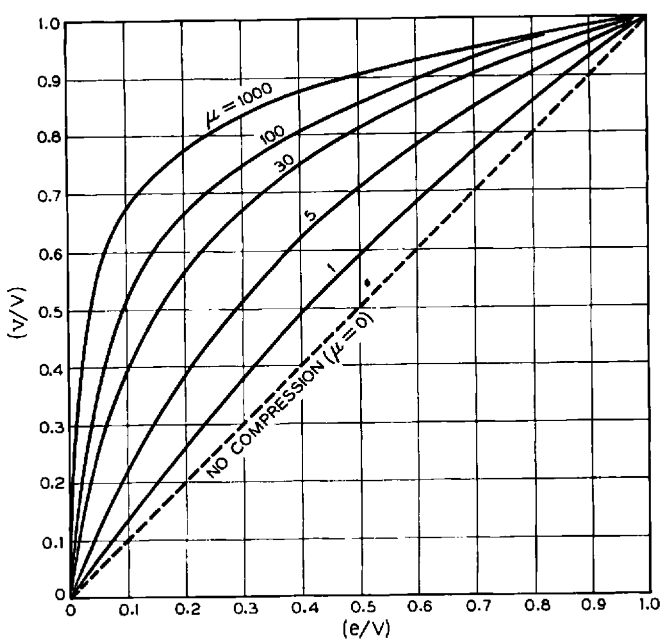

Fig. 3 — Typical logarithmic compression characteristics determined by equation (8a). The symmetrical negative portions, corresponding to equation (8b), are not shown.

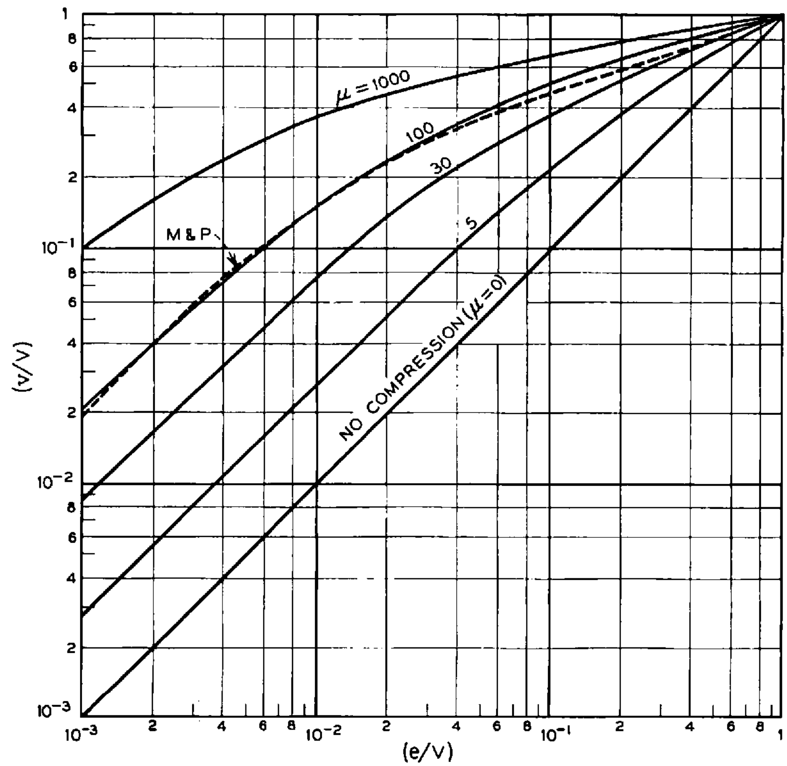

Fig. 4 — Logarithmic replot of compression curves shown in Fig. 3, to indicate detailed behavior for weak samples. The characteristic employed in the experiments of Meacham and Peterson[6] (M & P) is also shown. Similarity between this characteristic and the $\mu = 100$ curve testifies to the probable realizability of these logarithmic characteristics.

from consideration of the ratio of step size to corresponding pulse amplitude, $(\Delta e/e)$, since this quantity is a measure of the maximum fractional quantizing error imposed on individual samples. Hence the relation,

$$(e/\Delta e) = [N/2 \log (1 + \mu)](1 + V/\mu e)^{-1}$$

[which follows from (12a)] has been plotted, for $\mu = 10$, 100, and 1000, in Fig. 5. These curves reflect the fact that the sample to step size ratio reduces to the asymptotic forms:

$$(e/\Delta e) \rightarrow N/2 \log (1 + \mu) = \text{const} \qquad \text{for} \qquad (e/V) \gg \mu^{-1}$$

and

$$(e/\Delta e) \rightarrow [N/2 \log (1 + \mu)](\mu e/V) \qquad \text{for} \qquad (e/V) \ll \mu^{-1}$$

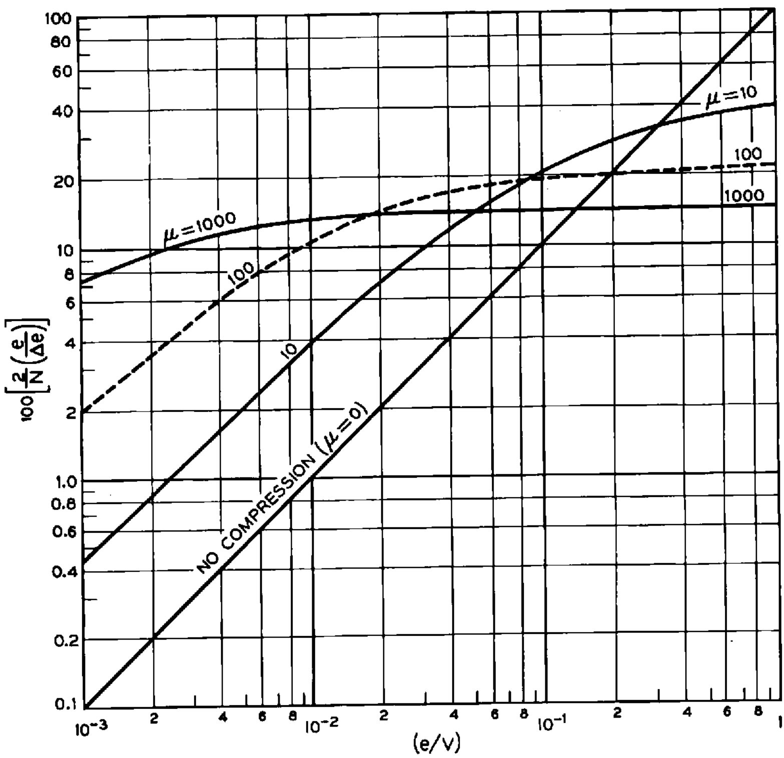

Fig. 5 — Pulse sample to step size ratios, as a function of relative sample amplitude, for various degrees of logarithmic companding (i.e., values of μ). The factor $(2/N)$ in the ordinate permits the curves to be drawn without reference to the total number of quantizing steps (N); the factor (100) is included to permit the ordinates directly to convey the proper order of magnitude for $(e/\Delta e)$, since present interest will be found to center about values of N for which $100(2/N) \sim 1$. As noted in the text, the ordinates, which constitute an index of the precision of quantization, approach constancy for $(e/V) \gg \mu^{-1}$, and vary linearly with abscissa for $(e/V) \ll \mu^{-1}$.

The essentially logarithmic behavior ($e/\Delta e \cong$ const) for large pulse amplitudes is intuitively desirable since it implies an approach to the equitable reproduction of the entire distribution of amplitudes in a specified signal. Although existing experimental evidence indicates that the small amplitudes are not only most numerous,[18] but also most significant for the *intelligibility*[21-23] of speech at constant volume, the absence of comparable evidence on the properties of *naturalness* makes it plausible to consider only those compression characteristics which give promise of providing the same, acceptably small, upper limit on the fractional quantizing error for pulse samples of all sizes.

For sufficiently small input pulses, $(e/\Delta e)$ becomes proportional to e, as a result of the linearity of the logarithmic function in (8) for small arguments. In view of our professed preference for logarithmic behavior, with $(e/\Delta e) \cong$ const., it is important to emphasize that the transition to linearity is not peculiar to (8), but is rather an example of the linearity to be expected of any suitably behaved (i.e., continuous, single-valued, with $(dv/de)_{e=0} > 1$) odd compression function, $v(e)$, capable of power series expansion, in the vicinity of the origin. In (8) this transition to linearity takes place where (e/V) is comparable to μ^{-1}. The extension of the region where $(e/\Delta e) \cong$ const. to lower and lower pulse amplitudes requires an increase in μ, and a concomitant reduction of the $(e/\Delta e)$ ratio for strong pulses.

Further evidence of the significance of the parameter μ may be deduced by evaluating the ratio of the largest to the smallest step size from the asymptotic expressions for $(e/\Delta e)$. Thus we find

$$\frac{(\Delta e)_{e=V}}{(\Delta e)_{e=0}} \to \mu \qquad \text{for} \qquad \mu \gg 1$$

which is a special form of the more general relation[5]

$$\frac{(\Delta e)_{e=V}}{(\Delta e)_{e=0}} = \frac{(dv/de)_{e=0}}{(dv/de)_{e=V}} = \mu + 1$$

which follows from our standard approximation of

$$(de/dv) \cong (\Delta v/\Delta e)$$

with $\Delta v =$ const.

B. *Comparison with Other Compandors*

An upper bound for companding improvement, which permits the quantitative evaluation of the penalty incurred (if any) through the restriction to logarithmic companding, is established in the Appendix. Comparison of the results to be derived from (8) with this upper bound will reveal that nonlogarithmic characteristics, which provide somewhat more companding improvement at certain volumes, are apt to prove too specialized for the common application to a broad volume range envisioned herein. The μ-characteristics do not suffer from this deficiency since the equitable treatment of large samples, which we have hitherto associated with an "intuitive naturalness conjecture," will be seen to tend to equalize the treatment of all signal volumes.

Finally, it will develop that (8), when applied to (6), has the added merit of calculational simplicity.

IV.* THE CALCULATION OF QUANTIZING ERROR

A. *Logarithmic Companding in the Absence of "DC Bias"*

As previously noted, we consider the effect of *uniformly* quantizing a *compressed* signal. If we designate the uniform output voltage step size by (Δv), then

$$(\Delta v) = \frac{2V}{N} \tag{10}$$

since the full voltage range between $-V$ and $+V$, of extent $2V$, is to be divided into N equal steps. For a number of levels, N, which is sufficiently large to justify the substitution of the differentials dv and de for the step sizes Δv and Δe, differentiation of (8a) yields

$$\frac{(\Delta v)}{V} = k\left[\frac{1}{1 + (\mu e/V)}\right]\frac{\mu(\Delta e)}{V} \tag{11}$$

where $k = 1/\log(1 + \mu)$.

Combining (10) with (11) and the counterpart of the latter in the domain of (8b), we find

$$\Delta e = \alpha(V + \mu e) \qquad \text{for} \qquad 0 \leqq e \leqq V \tag{12a}$$

and

$$\Delta e = \alpha(V - \mu e) \qquad \text{for} \qquad -V \leqq e \leqq 0 \tag{12b}$$

where

$$\alpha = 2 \log (1 + \mu)/\mu N \tag{13}$$

Substitution of (12) into (6) yields

$$\sigma = (\alpha^2/12)[V^2 + \mu^2\overline{e^2} + 2\mu V \overline{|e|}] \tag{14}$$

where the quantity $\overline{|e|}$ is introduced by the difference in sign in (12a) and (12b). For ordinary compandor applications, we may write

$$\overline{|e|} = 2\int_0^V eP(e)\, de \tag{15}$$

since the symmetry of the input signal provides that $P(-e) = P(e)$ and $\bar{e} = 0$.

---

* This passage contains mathematical details which may be omitted, in a first reading, without loss of continuity.

It is convenient to define the quantization error voltage ratio,

$$D = \frac{\text{RMS Error Voltage}}{\text{RMS Input Signal Voltage}} = (\sigma/\overline{e^2})^{\frac{1}{2}} \tag{16}$$

which takes the form

$$D = \log(1 + \mu)[1 + (C/\mu)^2 + 2AC/\mu]^{\frac{1}{2}}/\sqrt{3}N \tag{17}$$

when we define the quantities

$$A = \overline{|e|}/\sqrt{\overline{e^2}} = \frac{\text{Average Absolute Input Signal Voltage}}{\text{RMS Input Signal Voltage}} \tag{18}$$

and

$$C = V/\sqrt{\overline{e^2}} = \frac{\text{Compressor Overload Voltage}}{\text{RMS Input Signal Voltage}} \tag{19}$$

The simple linear proportionality of Δe to $(V \pm \mu e)$ results from the properties of the logarithmic function in differentiation. Other, seemingly more simple compression equations, when differentiated, yield much more complicated and unwieldy expressions for Δe. The value of this simplicity is evident in the absence, from (14), of moments of e higher than the second.

If we set $A = 0$, (17) reduces to one deduced by Panter and Dite[5]; their analysis erroneously associated A with $\bar{e} = 0$ rather than with $\overline{|e|}$, as a result of their tacit assumption that (12a) and (12b) are identical. They also imposed the restriction of considering only that class of input signals having peak values coincident with the compandor overload voltage, by defining V as the peak value of the signal in specifying C. The definition of C in terms of the independent properties of both signal $(\overline{e^2})$ and compandor (V) is then converted into one based solely on the properties of the signal. This interpretation leads to conclusions quite different from those to be presented here.

B. *Logarithmic Companding in the Presence of "DC Bias"*

It has heretofore been assumed that the input signal is symmetrically disposed about the zero voltage level since it may be expected that $\bar{e} = 0$ for speech. Although this is a standard assumption, subsequent discussion will disclose that it is probable, in actual practice, for the average value of the input signal to be introduced at a point other than the origin of the compression characteristic. In terms of Fig. 1, the signal is presented to the array of quantizing steps with its quiescent value displaced by an amount e_0 from the center of the voltage interval ($-V$ to $+V$).

Such an effect, regardless of its origin, may formally be described by considering the composite input voltage

$$E = e + e_0 \tag{20}$$

where e is the previously considered symmetrical speech signal and e_0 is the superimposed constant voltage.

Substitution of E for e in (8) and (12) yields

$$\sigma_E = (\alpha^2/12)[V^2 + \mu^2\overline{E^2} + 2\mu V\,\overline{|E|}] \tag{21}$$

where the subscript E is introduced to distinguish this result from (14). Note that the value $[e]_{E=0} = -e_0$ now separates the domain of applicability of (8a) and (12a) from that of (8b) and (12b), so that (15) is replaced by

$$\overline{|E|} = \int_{-V}^{-e_0} (-E)P(e)\,de + \int_{-e_0}^{V} EP(e)\,de \tag{22}$$

which reduces to

$$\overline{|E|} = \overline{|e|} + 2e_0 \int_0^{e_0} P(e)\,de - 2\int_0^{e_0} eP(e)\,de \tag{23}$$

Since $\bar{e} = 0$, and $e_0 = \text{const.}$, we also find

$$\overline{E^2} = \overline{e^2} + e_0^2 \tag{24}$$

C. *Application to Speech as Represented by a Negative Exponential Distribution of Amplitudes*

It is necessary to assume an explicit function for $P(e)$ in (15) and (23) before applying the general results which have thus far been deduced. We shall assume, as a simple but adequate first approximation, that the distribution of amplitudes in speech at constant volume[18] may be represented by

$$P(e) = G \exp(-\lambda e) \qquad \text{for} \qquad e \geqq 0 \tag{25}$$

where $P(-e) = P(e)$, $G = \lambda/2$, and $\lambda^2 = 2/\overline{e^2}$. The values of G and λ

follow from the standard relations

$$\int_{-\infty}^{\infty} P(e)\, de = 1$$

and

$$\int_{-\infty}^{\infty} e^2 P(e)\, de = \overline{e^2}$$

When applied to (15) and (18), with the upper limit in (15) replaced by ∞ with negligible error, (25) implies that

$$A = \overline{|e|}/\sqrt{\overline{e^2}} = 1/\sqrt{2} = 0.707 \tag{26}$$

Hence, (17) will be replaced, for numerical calculations, by the relation

$$\sqrt{3}ND = \log(1 + \mu)[1 + (C/\mu)^2 + \sqrt{2}C/\mu]^{\frac{1}{2}} \tag{27}$$

The corresponding substitution of (25) into (23) yields, for the case of $e_0 \neq 0$,

$$\overline{|E|} = e_0 + (\overline{e^2}/2)^{\frac{1}{2}} \exp(-\sqrt{2}C/B) \tag{28}$$

where we have introduced the "bias parameter,"

$$B = V/e_0 \tag{29}$$

When (28) is combined with (13), (21), and (24), we find, after some algebraic manipulation, that

$$\begin{aligned} \sqrt{3}ND_E &= \log(1 + \mu) \\ &\cdot [1 + (C/\mu)^2(1 + \mu/B)^2 + (\sqrt{2}C/\mu)\exp(-\sqrt{2}C/B)]^{\frac{1}{2}} \end{aligned} \tag{30}$$

where $D_E^2 = (\sigma_E/\overline{e^2})$. It is to be noted that D_E has been defined in terms of the ratio of σ_E to $\overline{e^2}$ rather than $\overline{E^2}$, so that

$$D_E^2 = \frac{\text{Mean Square Error Voltage}}{\text{Mean Square Speech Voltage}} \tag{31a}$$

$$= \frac{\text{Average Error Power}}{\text{Average Speech Power}} \tag{31b}$$

Examination of (30) reveals that it has the required property of reducing to (27) for $e_0 = 0$, i.e., for $B \to \infty$. Furthermore (27) and (30) indicate that $D_E \geqq D$ so that the addition of a dc component increases the quantizing error power when companding is used. The existence of such an impairment may easily be understood in terms of the physical interpretation of (6), as discussed in connection with Fig. 1.

Equations (27) and (30) also reveal that the penalty inflicted by a finite e_0 is largely determined by the ratio (μ/B). If $(\mu/B) \ll 1$, the presence of e_0 will be unimportant. At the other extreme, if $(\mu/B) \gg 1$, $(1 + \mu/B)^2 \rightarrow (\mu/B)^2$ and

$$\begin{aligned}\sqrt{3}ND_E \rightarrow \log\,(1 + \mu)& \\ \cdot[1 + (C/B)^2 + (\sqrt{2}C/\mu)\exp\,(-\sqrt{2}C/B)]^{\frac{1}{2}}&\end{aligned} \tag{32}$$

which proves to be relatively insensitive to changes in μ for the values of μ, C and B considered herein. In this case B largely usurps the algebraic role previously assigned to μ in (27).

D. *Uniform Quantization*: $\mu = 0$

The mean square quantization voltage error in the absence of companding, corresponding to direct, uniform quantization of the input signal, follows immediately from (7) and (10) since $\Delta v = \Delta e$ under these conditions. Thus

$$\sigma_0 = (\Delta v)^2/12 = V^2/3N^2$$

whence

$$D_0 = (\sigma_0/\overline{e^2})^{\frac{1}{2}} = C/\sqrt{3}N \tag{33}$$

This inverse proportionality of D_0 and N is well known.[2, 3, 5]

Equation (33) may also be deduced by letting μ approach zero in the expressions for D and D_E, since (8) implies that v approaches e as μ approaches zero. The fact that $D_0 = (D_E)_{\mu\rightarrow 0}$ reveals that, in the absence of companding, the addition of e_0 does not change the quantizing error power. This conclusion was anticipated in the discussion of Fig. 1.

V. DISCUSSION OF GENERAL RESULTS

Since the nature of a companded signal depends on a rather large number of variables, it is appropriate to consider their respective roles in general terms before discussing detailed system requirements. This general discussion will, however, emphasize those particular modes of operation which are suggested by existing proposals for application of PCM.[1, 3, 6] Thus, we shall consider common channel companding of speech in time division multiplex systems which employ binary number encoding.

A. *Quantitative Description of Conventional Operation* ($e_0 = 0$)

1. Number of Quantizing Steps (N)

The number of steps, N, is related to the choice of code. For a binary code, the relation takes the form, $N = 2^n$, where n is the number of binary digits per code group.

In the present discussion it will usually be convenient to regard n and N as fixed in order to permit comparison of various companding characteristics under the same coding conditions. Since both D and D_0 are inversely proportional to N, the quotient $(D/D_0)^2$, which is equal to the ratio of the quantizing error power in the presence of companding to that in the absence of companding, is independent of N. Consequently, as will be evident in the discussion of (37), the relative diminution of quantizing error (in db) afforded by companding is also independent of N.*

However, the value of N will determine the signal to quantizing error power ratio to which the companding improvement is to be added. Thus the number of digits per code group required for a particular application will ultimately be determined by the value of N which, in combination with suitable companding, will suffice to produce an acceptably low value of quantizing error power in relation to signal power.

2. Compandor Overload Voltage (V)

The compandor overload voltage, V, will be determined by the full load power objectives for the proposed system. Specifically, V will be equal to the amplitude of the sinsuoidal voltage corresponding to "full modulation."

3. Relative Signal Power (C)

The quantity $C = V/(\overline{e^2})^{\frac{1}{2}}$ will, for a given value of V, be determined by the rms signal voltage $(\overline{e^2})^{\frac{1}{2}}$.

The range of C values appropriate to a given system will therefore reflect the distribution of volumes to be encountered. In fact, the signal

* It must of course be understood that this independence requires a value of N sufficiently large to justify the approximations involved in the deduction of (27) and (33).

power in db *below* that corresponding to a full load sine wave is simply

$$10 \log_{10}\left[\frac{V^2/2}{\overline{e^2}}\right] = 10 \log_{10} [C^2/2] = 20 \log_{10} C - 3 \text{ } db \qquad (34)$$

It must be emphasized that C varies *inversely* with the rms signal voltage, so that weak signals are characterized by large values of C and strong signals by small values of C.

4. *Average Absolute Signal Amplitude* ($\overline{|e|}$)

The probability density of the signal manifests itself in the value assigned to the average absolute signal parameter, $A = \overline{|e|}/\sqrt{\overline{e^2}}$. This

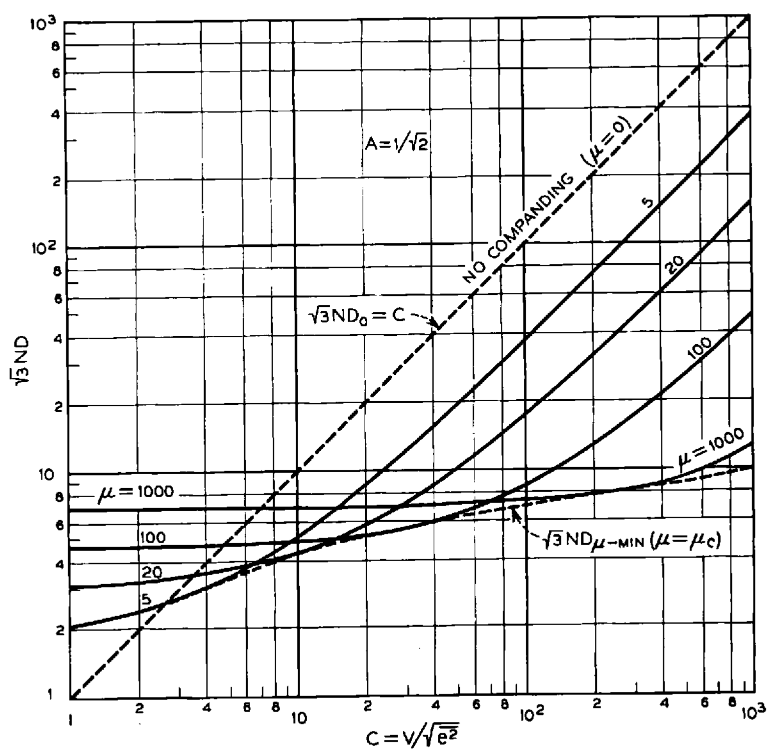

Fig. 6 — Variation of the rms error to signal voltage ratio (D) with relative signal strength, $C = V/\sqrt{\overline{e^2}}$, as given by equation (27) for various degrees of logarithmic companding.

quantity is a constant determined by the statistical properties of the class of signals being studied.

With the present choice of an exponential distribution of amplitudes to represent speech, [see (25)], we have seen that A takes on the value $1/\sqrt{2} = 0.707$. It develops that A is not very sensitive to the choice of $P(e)$, as may be judged by the values $\sqrt{2/\pi} = 0.798$, and $\sqrt{3}/2 = 0.866$ which would replace 0.707 if (25) were replaced by Gaussian and rectangular distributions, respectively. The value $A = 1/\sqrt{2}$ will be used in all numerical calculations; changes in the value of A to describe other classes of signals (e.g., the aforementioned Gaussian or rectangular distributions) will change the plotted results by no more than a fraction of a decibel.

5. *Degree of Compression* (μ)

From the foregoing it is clear that the essence of the compandor's behavior is embodied in the one remaining variable which appears in (8) and (17): the compression parameter μ.

The significance of μ has already received preliminary attention in connection with Figs. 3 to 5. Fig. 6, where comparison of behavior at constant N is facilitated by the choice of $\sqrt{3}ND$ as ordinate, exhibits the behavior of the ratio D as a function of C at constant μ. It will be observed that the curves in Fig. 6 do not extend below their common tangent which is labeled $\sqrt{3}ND_{\mu-\text{MIN}}$. The significance of this lower bound may be discussed in terms of Fig. 7 and the hypothetical ensemble of compandors to which we now direct our attention.

B. *Optimum Compandor Ensemble*

Consider the artificial situation in which our communication system includes an ensemble of instantaneous compandors, the members of which correspond to different values of μ in (8). Since companding improvement varies with signal strength, we permit ourselves the luxury of measuring the volume (i.e., C) of the input signal in order to assign the optimum degree of companding compatible with (8), to each individual signal. The compandor assigned to a signal is characterized by that particular value of the compression parameter, $\mu = \mu_c$, which is required to minimize D for a particular value of C. This critical compression parameter may be calculated from the requirement that

$$[\partial D/\partial \mu]_{A,C=\text{const}} = 0 \tag{35}$$

which yields

$$SC^2 + \mu_c A(S - 1)C - \mu_c^2 = 0 \tag{36*}$$

where $S = [(1 + \mu_c) \log (1 + \mu_c)/\mu_c] - 1$, when applied to (17).

The graph of (36) in Fig. 7 may be used to determine numerical values of μ_c without repeated recourse to the equation.

The curve labeled $\sqrt{3}ND_{\mu\text{-MIN}}$ in Fig. 6 was determined by substituting values of μ_c, obtained from Fig. 7, into (27). Each curve in Fig. 6

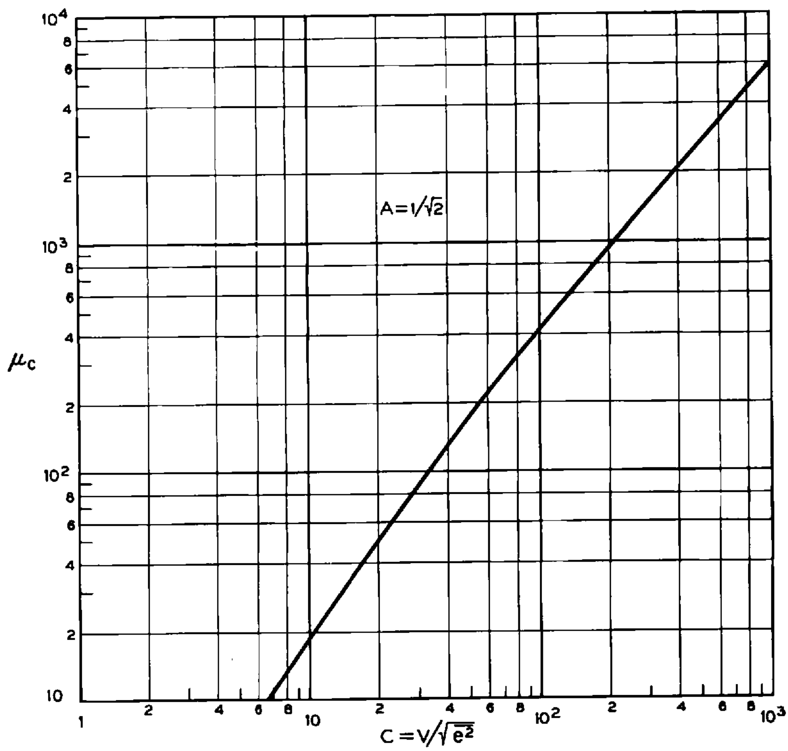

Fig. 7 — Critical compression parameters, μ_c, required to minimize the quantizing error power as a function of relative signal strength, as determined by equation (36). Each point on the curve defines a compandor in the optimum compandor ensemble. It must be understood that such an ensemble provides the best performance consistent with equation (8) rather than the absolute minimum quantizing error discussed in the Appendix.

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* A similar equation, with $A = 0$ corresponding to the previously noted erroneous identification of A with $\bar{e}$ rather than $\overline{|e|}$, has been deduced by Panter and Dite.[5] Their definition of C as a "crest factor" changes the significance of what we have called $D_{\mu\text{-MIN}}$ and does not lead to the ensemble interpretation of μ_c.

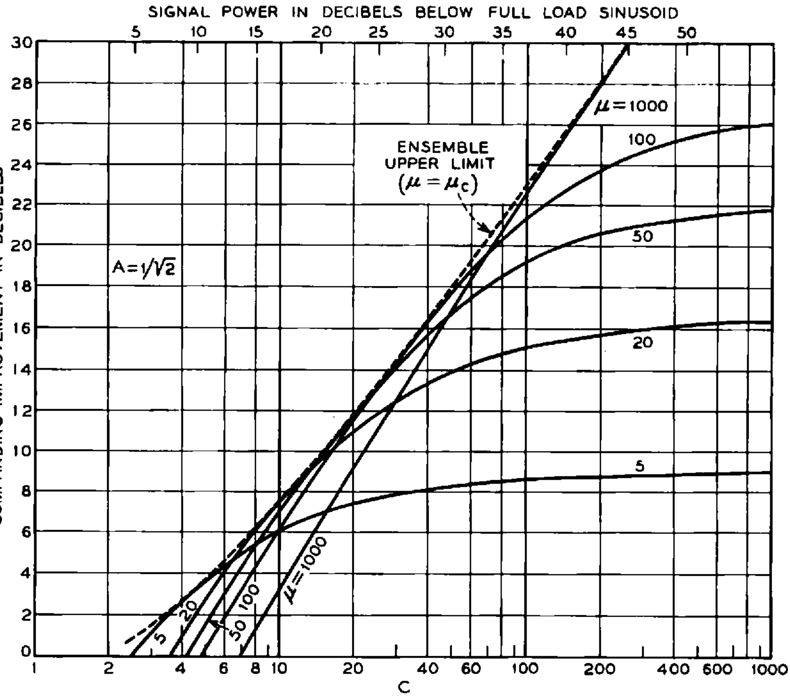

Fig. 8 — Companding improvement (in db), as calculated from equation (37), for various values of μ. The saturated improvement for weak signals (relatively constant ordinate for large C values) is identical with the asymptotic behavior for weak signals which is predicted by Fig. 9.

is tangent to this lower bound at the single value of C which corresponds to $\mu = \mu_c$.

In conventional systems, a single common channel compandor,[6] characterized by a single value of μ, is substituted for the optimum ensemble. Although $D_{\mu-\mathrm{MIN}}$ is then attainable at only one value of C, it is instructive to compare each value of D with the corresponding value of $D_{\mu-\mathrm{MIN}}$. Indeed, consideration of the optimum ensemble has, in one sense, reduced the problem of choosing an appropriate μ for a given application to the choice of that particular value of C at which equality of D and $D_{\mu-\mathrm{MIN}}$ is desired.

In Fig. 6, the line representing performance in the absence of companding corresponds to (33) for D_0. D_0 and $D_{\mu-\mathrm{MIN}}$ are seen to be similar for strong signals (low values of C). Furthermore, it is important to note that D_0 does not constitute an upper bound for D; thus the companding weakest signals, regardless of the shape of the characteristic in the nonlinear region. Equation (8) implies

$$(v/e)_{e\to 0} = (\Delta v/\Delta e)_{e\to 0} = \mu/\log(1+\mu)$$

so that for each value of μ, the constant companding improvement for the weakest signals is

$$20 \log_{10} [\mu/\log(1+\mu)] \text{ db}$$

which is plotted in Fig. 9. Fig. 8 supplements Fig. 9 in revealing the actual volumes required for the realization of this weak signal saturation, as well as the detailed behavior for stronger signals.

D. *Companding Improvement for $e_0 \neq 0$*

Since we will usually regard a nonzero value of e_0 as an undesirable

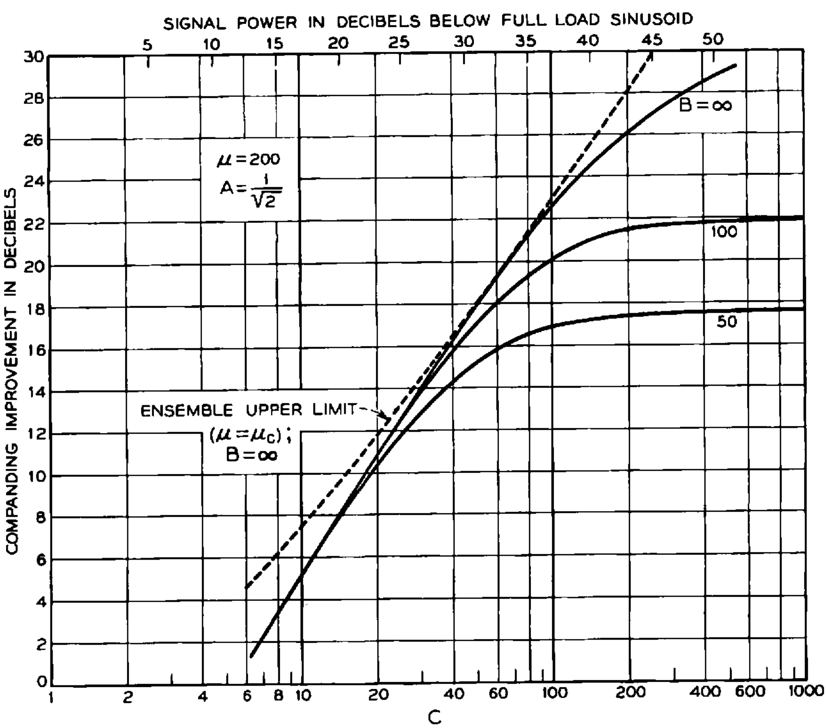

Fig. 11 — The effect of a "dc component" on companding improvement for $\mu = 200$. For further details, see the caption of Fig. 10.

perturbation, we wish to study the modification of companding improvement produced by the introduction of a finite value of $B = V/e_0$, i.e., substitution of (30) for (27), when N, V, C, A, and μ remain unchanged.

In Figs. 10 to 13 we have replotted the companding improvement curves shown in Fig. 8 for $\mu = 100$, 200, 500, and 1,000, respectively. These curves correspond to $B = \infty$. The difference between these curves and those for finite values of B in Figs. 10 to 13 is the impairment (in db) inflicted by the presence of e_0. This impairment may be appreciable for weak signals (large C). As already noted in connection with (32) the impairment is not severe for $(\mu/B) \ll 1$. Furthermore, the appropriation of the algebraic role of μ by B, when $B \ll \mu$, which was previously noted in (32), manifests itself in the striking similarity of the weak signal behavior of all the curves for $B = 50$ and 100 in Figs. 10 to 13.

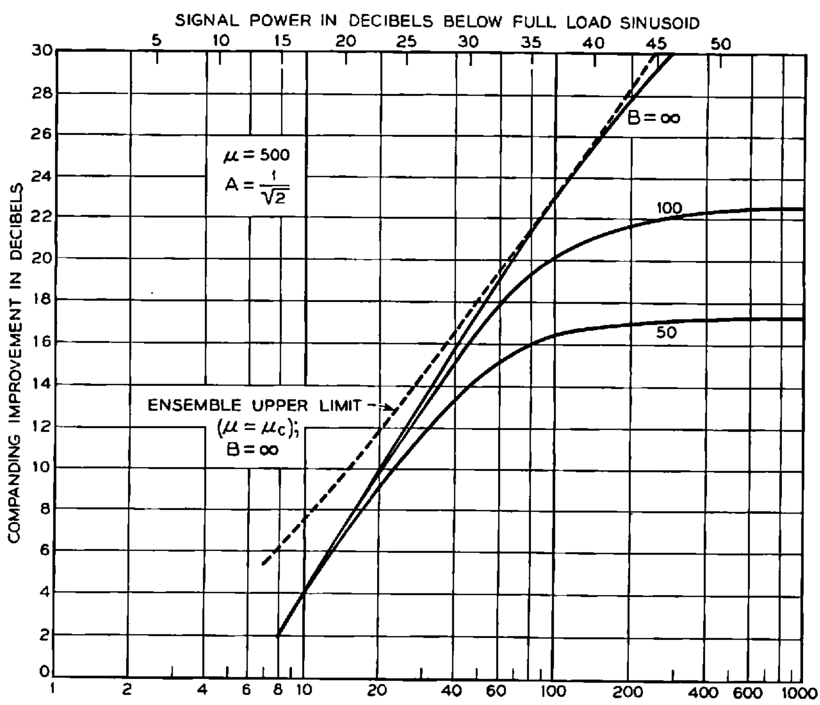

Fig. 12 — The effect of a "dc component" on companding improvement for $\mu = 500$. For further details, see the caption of Fig. 10.

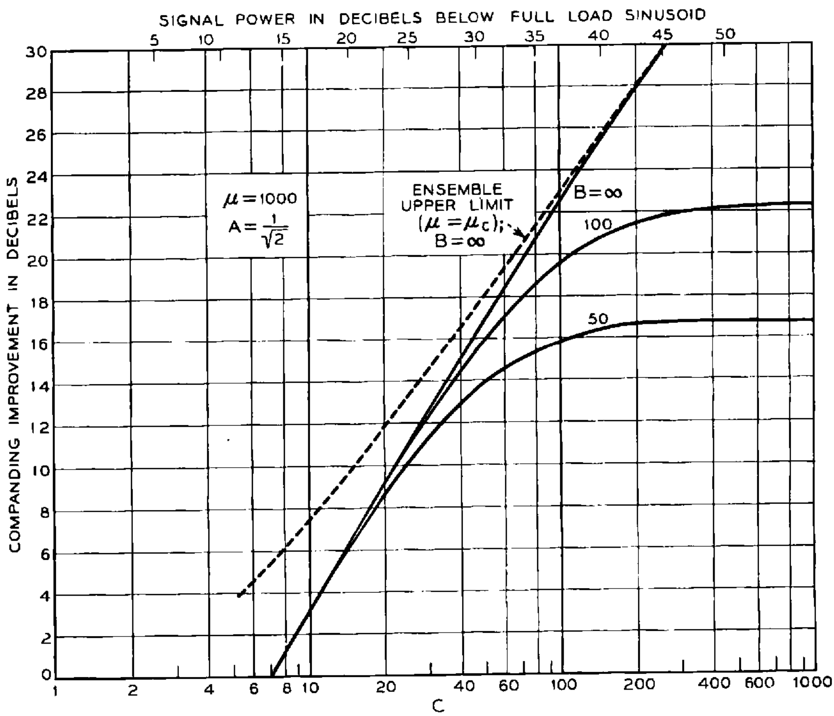

Fig. 13 — The effect of a "dc component" on companding improvement for $\mu = 1{,}000$. For further details, see the caption of Fig. 10.

VI. APPLICATION OF RESULTS TO A HYPOTHETICAL PCM SYSTEM

Consider the application of these results to the planning of a typical, albeit hypothetical, communication system.

A. *Speech Volumes*

Suppose it is desired to transmit signals covering a 40 db power range, with the strongest and weakest speech volumes each separated by 20 db from the average anticipated signal power at the compressor input.*

The strongest signal power is then used to determine the value of the compandor overload voltage, V. In this case a value of V corresponding to a full load sine wave 10 db above the loudest signal, [see (34)], appears adequate.* Although this choice may at first appear arbitrary,

* These values are sufficiently close to those cited as representative by Feldman and Bennett, in connection with Fig. 2 of Reference 11, to be considered quite realistic.

the value of 10 db is probably no more than a few db removed from the value which would be chosen in any efficient application of PCM to quality telephony. It results from the need to balance the requirement of a value of V sufficiently high to avoid intolerable clipping of the peaks of the loudest signals against the obvious advantage of reducing the quantizing step size by minimizing the voltage range to be quantized. We have neglected clipping in our calculations since it was assumed that the significant peaks of the loudest signals should not exceed V for quality telephony. Existing information on clipped speech[21-23] and one digit PCM[10] indicates that the clipping impairment we seek to avoid is largely one of loss of naturalness rather than reduction in intelligibility. The choice of a maximum volume 10 db below a sinusoid of amplitude V implies that speech peaks 13 db, [see (34)], above the maximum rms signal voltage are being ignored, which appears reasonable in the light of available experimental evidence.[18, 22]

It follows from these assumptions that the average and weakest signals are respectively 30 db and 50 db below full sinusoidal modulation.

B. *Choice of Compression Characteristic*

1. Ideal Behavior for Speech

If we adopt the aim of achieving the smallest over-all departure from the ensemble limit of improvement, it seems reasonable to choose that compandor in the optimum ensemble which corresponds to average speech ($C \cong 45$). This requirement, in conjunction with Fig. 7, establishes a lower bound of about 150 for μ. The significance of this choice may be clarified by reference to Fig. 14, which depicts departures from the optimum ensemble limit of improvement, resulting from restriction to a single value of μ for all volumes.

The corresponding upper bound will be determined by the alternative of furnishing optimum improvement to the weakest signals ($C \cong 450$) in spite of the concomitant impairment of loud speech. Reference to Fig. 7 then dictates a choice of μ in the vicinity of 2,500. From Fig. 6 it is clear that this value implies that D is essentially constant and independent of C throughout the range of interest. Appreciably larger values of μ would actually lead to the undesirable extreme of $D > D_{\mu\text{-MIN}}$ for all signals under consideration.

We therefore conclude that attention may profitably be confined to the interval $150 \lesssim \mu \lesssim 2{,}500$, the magnitude of which is adequately conveyed by the simple expression

$$100 \lesssim \mu \lesssim 1{,}000 \qquad (38)$$

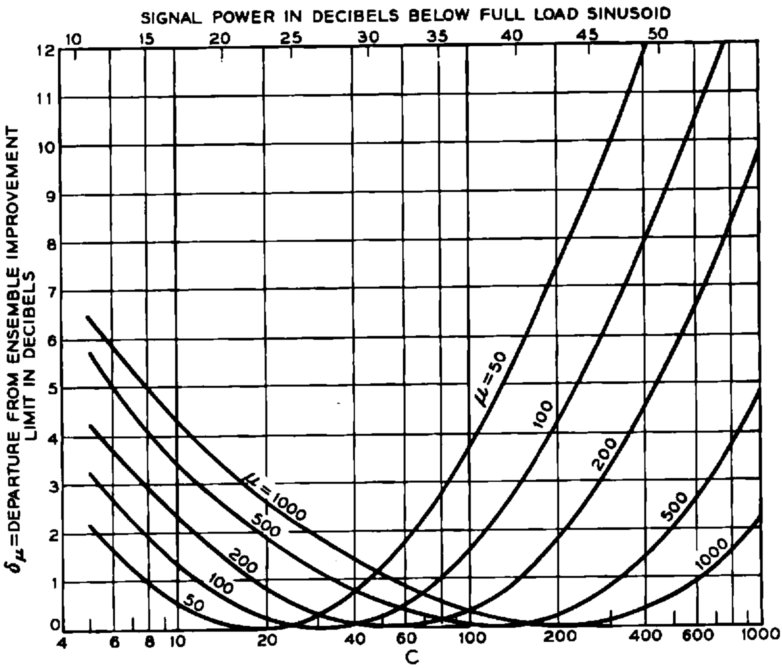

Fig. 14 — Improvement selectivity, i.e., departures from the ensemble upper limit of improvement due to the use of a single value of μ rather than the ensemble of μ_c values. The minima at $\delta_\mu = 0$ locate the signals (C) for which μ and μ_c coincide. All curves correspond to the case where the "dc component", e_0, is zero.

Lest it appear that this range is so broad as to offer very little practical guidance, it should be noted that (38) defines a rather narrow range of characteristics in Figs. 3 and 4. The assumption that this range may be realized in practice appears reasonable in view of the similarity to the characteristic actually used by Meacham and Peterson, which is shown in Fig. 4.

2. Practical Limitations on Companding Improvement

(*a*) *Mismatch Between Zero Levels of Signal and Compandor.* Although the present discussion has hitherto been confined to ideal compandor action, it lends itself quite naturally to the analysis of a significant departure from ideal behavior which may be expected to result from the use of an instantaneous compandor on a common channel basis in time division multiplex systems.

It will probably be impractical to balance the channel gating circuits (required to provide sequential connection of individual channels to a single compressor)[1, 6] sufficiently to guarantee exact coincidence of the average input signal ($\bar{e} = 0$) in each channel and the center of the $-V$ to $+V$ voltage range ($e = 0$) presented by the compressor. Thus the input, e, would appear to the compressor in the form $E = e + e_0$. The consequences of the appearance of the undesirable constant term, e_0, may be inferred from study of Figs. 10 to 13 and (30).

We shall assume that, owing to the present state of gating technology, $B = V/e_0$ may reasonably be expected to assume values in the range $100 \lesssim B \lesssim 1{,}000$.

For companding corresponding to $100 \lesssim \mu \lesssim 1{,}000$, Figs. 10 to 13 indicate that, if B can be confined to the vicinity of 1,000, the departure from the ideal behavior corresponding to $B = \infty$ will be virtually negligible.

However, should it prove necessary to work with $B \cong 100$, it is clear from Figs. 10 to 13 that the companding improvement for weak signals would be relatively independent of μ in the interval $100 \lesssim \mu \lesssim 1{,}000$ (with a saturation value of about 20.5–22.5 db).* In this event, compression to a degree greater than that represented by $\mu = 100$ would provide less improvement for strong and average speech without the compensation of significantly greater improvement for weak signals. Reduction of μ below 100 would not be fruitful since the sensitivity of companding improvement to changes in μ is restored for values satisfying the condition of $(\mu/B) = (\mu/100) < 1$.

The significance of the values $e_0 \sim V/1{,}000$ and $V/100$ may perhaps better be appreciated in terms of a comparison of e_0 with the weakest signals under consideration. Since $(B/C) = \sqrt{\overline{e^2}}/e_0$, a signal to dc bias power ratio may be calculated, in db, from the expression $20 \log_{10} (B/C)$. For the weakest signals under consideration ($C \sim 400$), the values $B = 1{,}000$ and 100 correspond respectively to $(\sqrt{\overline{e^2}}/e_0) = 2.5$ and 0.25, or to signal to dc bias power ratios of $+8$ db and -12 db. Thus, for the hypothetical system now under study, the value of e_0 becomes significant (roughly) when it exceeds the weakest rms signal.

Actually e_0 would be expected to vary with time for a given channel and to vary from channel to channel at any instant. On the assumption that $|e_0| = V/100$ (i.e., $B = 100$) will constitute the upper bound of such variations, the companding improvement corresponding to a particular value of μ must now be specified in terms of the region between the $B = \infty$ and $B = 100$ curves in Figs. 10 to 13, rather than by reference to a single value of B and its corresponding curve. Since the lower

* This corresponds to the behavior of D_E for $(\mu/B) \gg 1$ which was noted in the discussion of (30). In this connection, see the discussion of Fig. 19.

bounds of all these regions (see Figs. 10 to 13) are approximately coincident (for $100 \lesssim \mu \lesssim 1{,}000$), the advantage of increasing μ substantially beyond 100 will depend largely on the expectation of encountering values of $e_0 \rightarrow (V/1{,}000)$ with sufficient frequency in the various channels served by the common compressor.

These arguments may of course be applied, with suitable modifications depending on the range of C, μ, and B values requiring attention, to any effect capable of formal description in terms of an effective dc bias superimposed on the signal input to the compressor.

(*b*) *Background Noise Level.* It does not seem reasonable to strive for an increase of the signal to quantizing error power ratio substantially beyond that value which is subjectively equivalent to the anticipated ratio of signal to background noise from other sources.

Since the quantizing error power depends on the number of digits per code group, the comparison of quantizing error power and noise power is reserved for subsequent discussion of the required number of quantizing steps. It will be noted that the comparison must remain somewhat speculative in the absence of a determination of the subjective equivalence of quantizing error power and noise.

C. *Choice of the Number of Digits Per Code Group*

1. *Ideal Behavior for Speech*

As previously remarked, the number of quantizing steps will determine the ratio of signal to quantizing error power to which the companding improvement is to be added. Since the quantizing error power is inversely proportional to $N^2 = 2^{2n}$, this power will be reduced by 6 db for each additional digit. Comparison of this 6 db per digit improvement with the roughly 24 to 35 db improvement corresponding to weak signals in Fig. 8 (for $100 \lesssim \mu \lesssim 1{,}000$) reveals that, *for such signals, companding is equivalent to the addition of four to six digits per code group, i.e., to an increase in the number of quantizing steps by a factor between* $2^4 = 16$ *and* $2^6 = 64$. This equivalence is portrayed in Fig. 9. Our failure to realize a companding improvement of about 43 db as predicted for $\mu = 1{,}000$ in Fig. 9 may be traced to the fact that the weakest signals now under consideration are not sufficiently weak to be confined to the linear region $(e/V) \ll \mu^{-1})$ of the $\mu = 1{,}000$ characteristic. This is reflected in the unsaturated improvement exhibited in Fig. 8 for the weakest signals when $\mu = 1{,}000$.

Although it is clearly preferable to suppress quantizing error power by companding rather than by increasing the number of quantizing

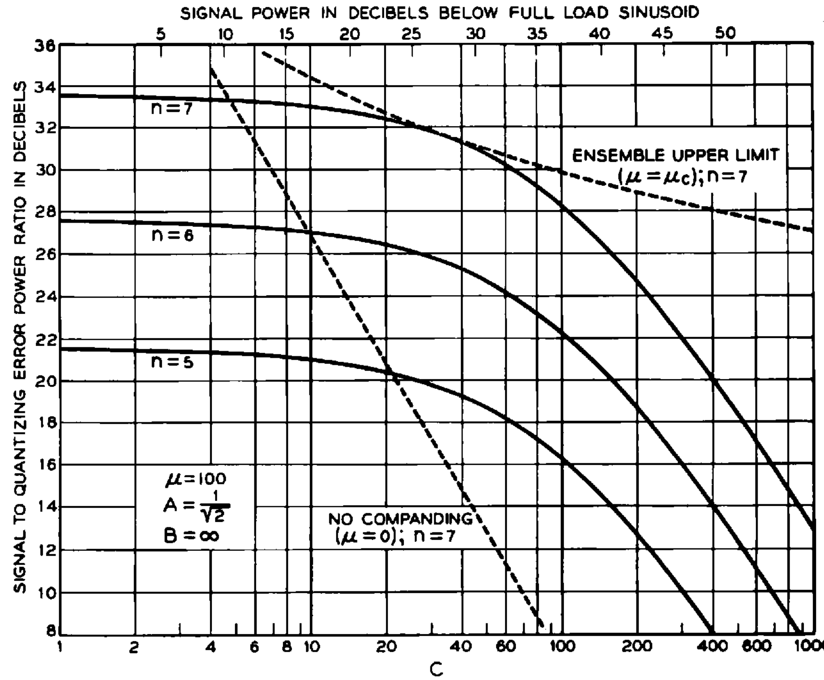

Fig. 15 — Signal to quantizing error power ratios (calculated, in db, from equation (39)) as a function of relative signal power for companding corresponding to $\mu = 100$. Curves are shown for $n = 5, 6$, and 7 digits per code group. For comparison, the results for seven digits in the absence of companding ($\mu = 0$) as well as for the ensemble upper limit ($\mu = \mu_c$) are included. $B = \infty$ throughout.

steps, it is apparent that the upper limit of companding improvement will set a lower limit on the number of digits required for satisfactory operation.

Once again we begin with the consideration of pure speech signals. The expression

$$\begin{aligned} -10 \log_{10} (D^2) &= -20 \log_{10} D \\ &= \text{Signal to Quantizing Error Power Ratio in db} \end{aligned} \tag{39}$$

has been plotted against C in Figs. 15 to 18 for $\mu = 100, 200, 500$, and 1,000 respectively. In each case the behavior for 5, 6, and 7 digits is compared with the extremes of $\mu = 0$ (no companding) and $\mu = \mu_c$ (ensemble upper limit) for 7 digits.

It must be conceded at the outset that experimental work is required to formulate standards for quantizing error power similar to those

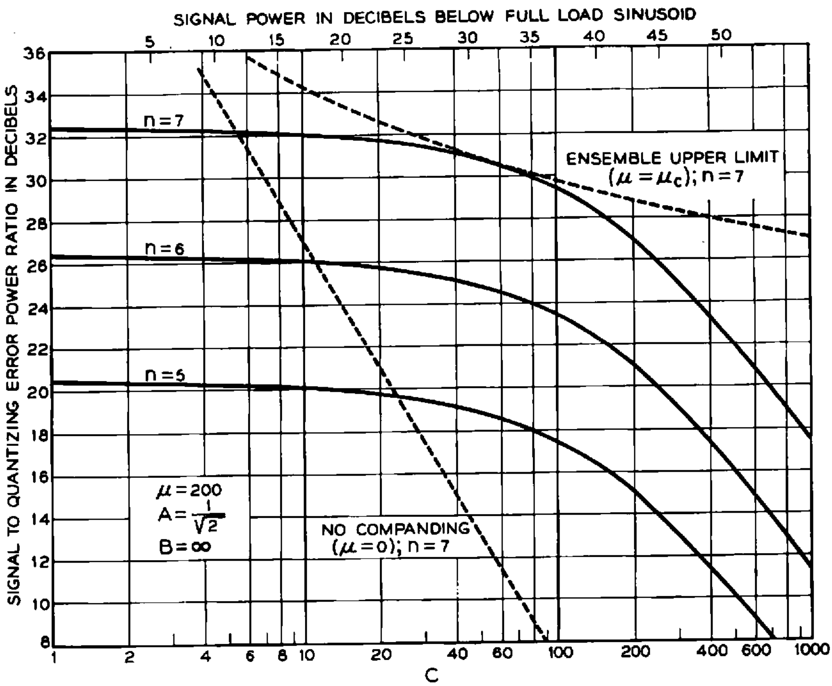

Fig. 16 — Signal to quantizing error power ratios (in db) as a function of relative signal power for companding corresponding to $\mu = 200$. Symbols have the same significance as in Fig. 15. $B = \infty$ throughout.

which have been established for conventional noise and distortion. If these were available, graphs such as those in Figs. 15 to 18 could be used to select the proper number of digits to be used with various degrees of compression. In the absence of such information we shall complete this illustrative study by adopting a signal to quantizing error power ratio of at least 20 db as a tentative standard of adequate performance at all volumes.*

Figs. 15 and 16 show that seven digits (i.e., $2^7 = 128$ tapered quantizing steps) and $\mu \cong 150$ will meet this objective. Furthermore Figs. 17 and 18 indicate that six digits ($2^6 = 64$ tapered steps) would suffice provided (38) is replaced by the more stringent limitation,

$$500 \lesssim \mu \lesssim 1{,}000 \tag{40}$$

* This value does not appear unreasonable, as a first approximation, in terms of experience with noise and harmonic distortion.

2. Practical Limitations

(*a*) *Mismatch Between Zero Levels of Signal and Compandor.* From the previous discussion of the effect of e_0 on the choice of μ, it is clear that, if B can be confined to the vicinity of 1,000, the analysis of the required number of digits in the absence of instability ($B = \infty$) may be applied.

On the other hand, behavior for $B = 100$ may be judged from the plot of signal to quantizing error power ratio versus signal power for $\mu = 100$ and 1,000 (with seven digits) shown in Fig. 19. Since this ratio now fails to exceed about 16 db for the weakest signals of interest, we conclude that an increase to eight digits ($2^8 = 256$ tapered steps), with a concomitant 6 db improvement for all signals, is required to meet our 20 db objective. These curves also illustrate the previously noted meager improvement for weak speech which accompanies the increase from $\mu = 100$ to 1,000 when $B = 100$. Actually, an optimum solution is attained for an intermediate value of μ, but the advantage is too small to be of interest (see Figs. 10 to 13).

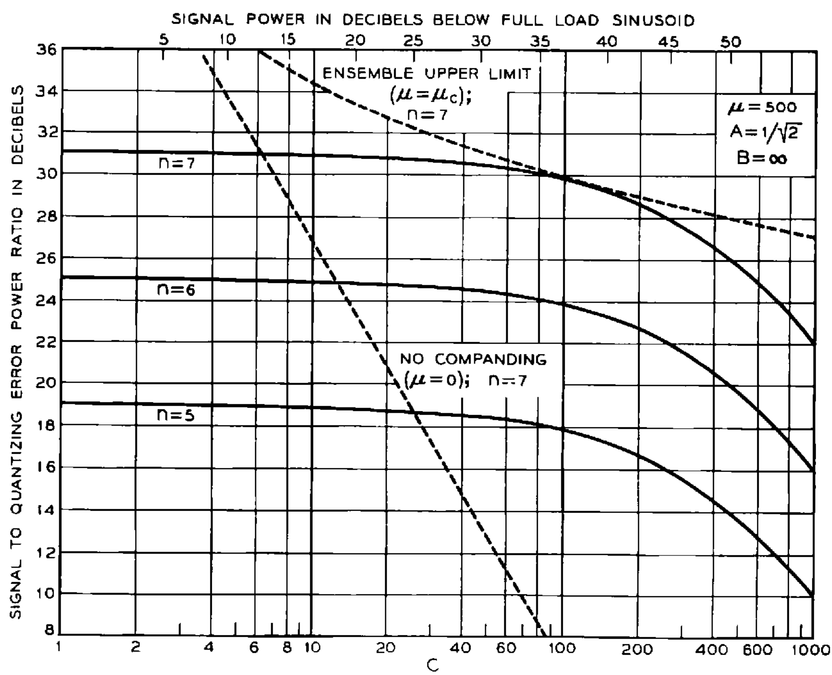

Fig. 17 — Signal to quantizing error power ratios (in db) as a function of relative signal power for companding corresponding to $\mu = 500$. Symbols have the same significance as in Fig. 15. $B = \infty$ throughout.

The recognition that use of $B = 100$ rather than a value approaching 1,000 may imply a change from six to eight digits per code group (e.g., for $\mu = 1{,}000$), representing an increase of 33 per cent in the required bandwidth in the transmission medium as well as a significant increase in the complexity of the multiplex terminal equipment, provides the proper perspective for competent appraisal of the cost of improving gate circuitry to the point where B would approach 1,000. These considerations might be of crucial importance in the planning of actual PCM systems.

Finally these results also show that caution is required in attempting to determine an adequate number of digits and/or degree of compression from listening tests employing preliminary experimental equipment. If the conditions of the test do not duplicate exactly the expected behavior of the channel gates to be used in the final system, the transition from the laboratory to practice might lead to an embarrassing disappearance

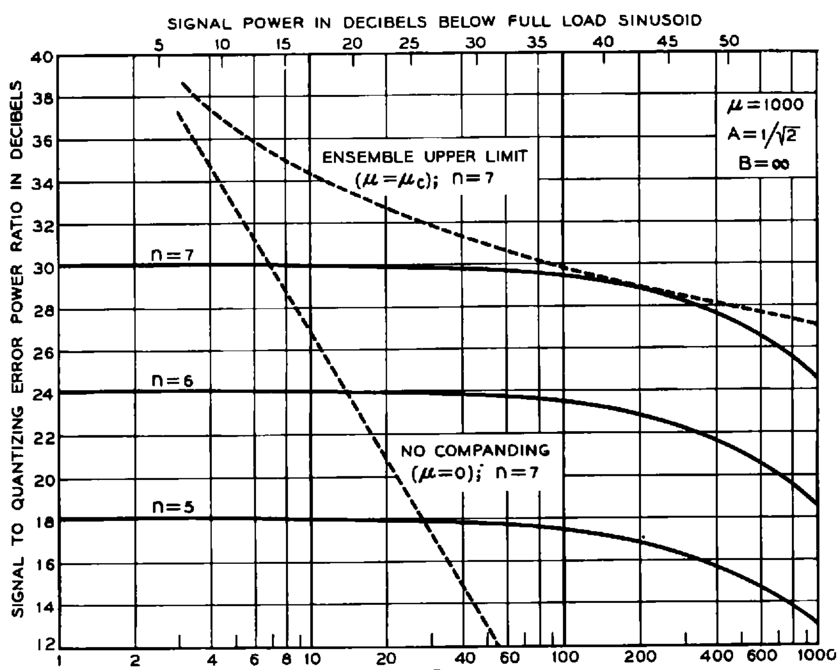

Fig. 18 — Signal to quantizing error power ratios (in db) as a function of relative signal power for companding corresponding to $\mu = 1{,}000$. Symbols have the same significance as in Fig. 15. $B = \infty$ throughout.

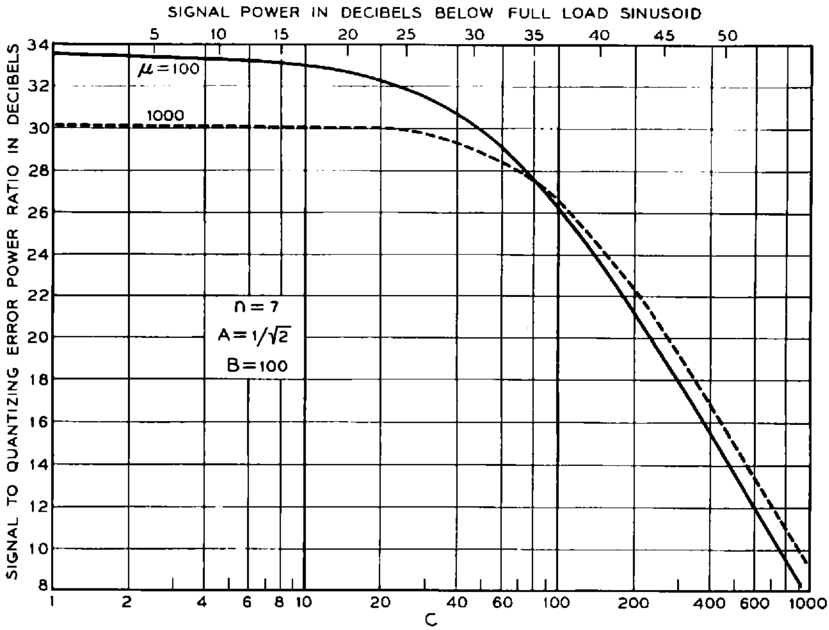

Fig. 19 — Signal to quantizing error power ratios (in db) as a function of relative signal power for companding corresponding to μ = 100 and 1,000 when n = 7 digits per code group and a dc component corresponding to B = 100 is present in the signal. The influence of the dc component may be judged by comparing these curves with those shown in Figs. 15 and 18 for n = 7. Corresponding results for different values of n may be derived by the addition or subtraction of appropriate multiples of 6 db from each ordinate.

of virtually all the anticipated companding improvement for weak signals.

(*b*) *Background Noise Level.* We have already noted the probable futility of increasing the signal to quantizing error power ratio considerably beyond that value which is subjectively equivalent to the anticipated ratio of signal to background noise from other sources.

If the subjective relation between quantizing error power and noise power were known, the curves in Figs. 15 to 19 could be redrawn for meaningful comparison with ratios of signal to background noise. In the absence of such information, we shall assume as a first approximation, that noise and quantizing error power are directly comparable.*

Suppose that we set an upper limit on the background noise by con-

* The similarity between noise and quantizing error power has often been noted. For example, one may consult references 2, 6 and 12 as well as Appendix I on "Noise in PCM Circuits" in Reference 11. The assumption of direct comparability is also to be found in Reference 4.

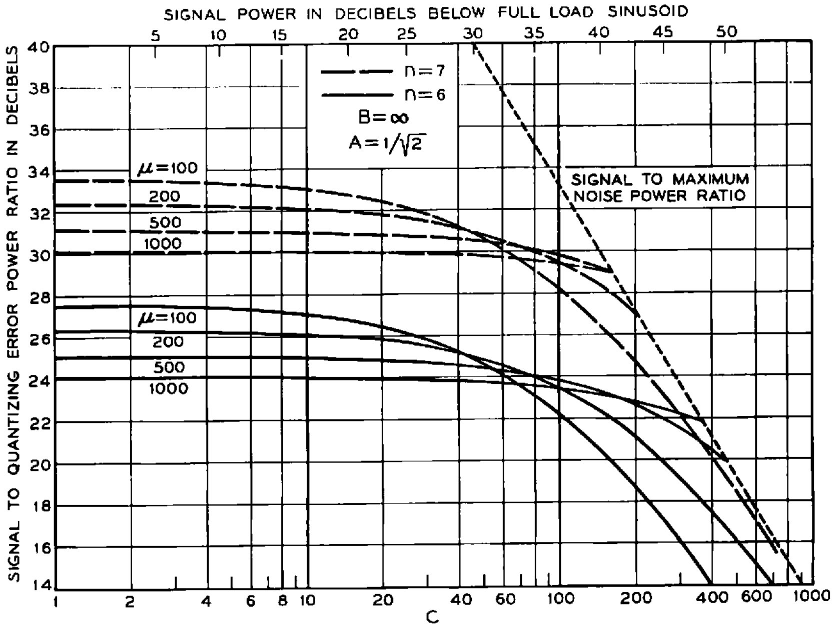

Fig. 20 — Curves illustrating the comparison of signal to quantizing error power ratios with the ratio of signal to background noise. The line representing the signal to maximum noise ratios corresponds to the hypothetical case where the maximum background noise is determined by the requirement that the signal to noise ratio be 20 db for a signal 50 db below full sinusoidal modulation.

sidering a value providing a signal to noise ratio of 20 db for the weakest signals in our hypothetical system. A signal to maximum noise power curve may then be drawn as a function of signal power for this constant value of noise power. Such a graph has been combined, in Fig. 20, with curves such as those which have previously appeared in Figs. 15 to 19. These curves have been terminated at their intersections with the line representing the signal to maximum noise power ratio since we are assuming that little benefit will be derived from a signal to quantizing error power ratio in excess of the signal to maximum noise power ratio.

From Fig. 20 it is apparent that the previous conclusions that six and seven digits are worthy of consideration are unaffected by the stipulation that the signal to quantizing error power ratio should not greatly exceed the signal to maximum noise power ratio. Similarly, the conclusions based on Fig. 19 (for $B = 100$) remain unchanged since the curves therein fall below the maximum noise curve of Fig. 20 for all values of the abscissa.

D. *Possibility of Using Automatic Volume Regulation*

The realization that the quantizing impairment experienced by weak signals in the absence of compression stems from their inability to excite a sufficient number of the quantizing steps which must be provided to accommodate loud signals, leads directly to the suggestion that automatic volume regulation be used to permit all signals to be "loud," i.e., to excite the entire aggregation of quantizing steps. In its simplest form, this would be accomplished by automatic amplification of the long time average speech power in each channel to provide a constant volume input to the common channel equipment.

Study of the present results indicates that if all signals were of constant volume, about 10 to 15 db below full sinusoidal modulation (to provide an adequate peak-clipping margin), satisfactory operation, corresponding to signal to quantizing error power ratios in excess of 20 db, might be achieved *without companding* by using as few as five or six digits per code group. In evaluating this alternative, the advantages of reduction of bandwidth, decreased complexity of quantizing and coding equipment, and elimination of the common channel compandor, must be balanced against the disadvantage of providing separate volume regulators in each channel.

E. *Comparison with Previous Experimental Results*

The literature contains seemingly contradictory statements about whether five,[17] six,[14] or seven[6, 7] digits per code group are required for satisfactory performance in speech listening tests. Evaluation of these conclusions is frequently hampered by the lack of specification of either the degree of companding employed or the range of speech volumes requiring transmission. Different conclusions may therefore be consistent, inasmuch as the systems may differ significantly in the required volume range, degree of companding, size of the "effective dc component" in the signal, and even in the subjective standards used to judge performance.

Fortunately, the description of an experimental toll quality system by Meacham and Peterson[6] is sufficiently detailed to permit some comparison. The range of volumes they considered suggests that direct comparison with our hypothetical system is fairly reasonable. Their empirical choice of seven digits, with a compression characteristic virtually indistinguishable from that corresponding to $\mu = 100$ (see Fig. 4) is in excellent agreement with the present conclusions.

Furthermore, the conclusion that five or six digits, without companding, might be employed in conjunction with volume regulation is completely consistent with Goodall's experimental results.[10]

VII. CONCLUSIONS

An effective process for choosing the proper combination of the number of digits per code group and companding characteristic for quantized speech communication systems has been formulated. Under typical conditions, the calculated companding improvement *for the weakest signals* proves to be equivalent to the addition of about 4 to 6 digits per code group, i.e., to an increase in the number of quantizing steps by a factor between $2^4 = 16$ and $2^6 = 64$.

Although a precise application of the results requires a more detailed knowledge of the subjective nature of the quantizing impairment of speech than is presently available, the assumption of reasonably typical system requirements yields conclusions in good agreement with existing experimental evidence.

ACKNOWLEDGMENTS

Frequent references in the text attest to the indebtedness of the author to the writings of Bennett and Panter and Dite. It is also a pleasure to acknowledge stimulating conversations on certain aspects of the problem with J. L. Glaser, D. F. Hoth, B. McMillan, and S. O. Rice.

APPENDIX

THE MINIMIZATION OF QUANTIZING ERROR POWER

In spite of the demonstrated utility of the μ-characteristics, one cannot avoid speculating about the possibility of achieving substantially more companding improvement by using a characteristic which differs from (8). We shall therefore outline a study of the actual minimization of quantizing error power without regard to the relative treatment of various amplitudes in the signal. The results will confirm that a significant reduction of the quantizing error power beyond that attainable with logarithmic companding is self-defeating — for it not only imposes the risk of diminished naturalness, but also implies a compandor too "volume-selective" for the applications envisioned herein.

1. The Variational Problem and Its Formal Solution

Equation (6) may be expressed in the form

$$\sigma = \frac{2V^2}{3N^2} \int_0^V (dv/de)^{-2} P(e)\, de \tag{A-1}$$

where $P(e)$ has been assumed to be an even function. The function, $v(e)$, which will minimize (A-1), subject to the usual boundary conditions at $e = 0$ and $e = V$, may be obtained by solving the Euler differential equation of the variational problem.[28] For (A-1), this takes the form

$$(dv/de) = KP^{1/3} \tag{A-2}$$

where the constant K is given by

$$K = V \Big/ \int_0^V P^{1/3}\, de \tag{A-3}$$

Hence the minimum quantizing error is given by

$$\sigma_{\text{MIN}} = 2\left[\int_0^V P^{1/3}\, de\right]^3 \Big/ 3N^2 \tag{A-4}^*$$

2. Representation of Speech by an Exponential Distribution of Amplitudes

We shall assume, as in (25), that the distribution of amplitudes in speech at constant volume[18] may be represented by

$$P(e) = G \exp(-\lambda e) \text{ for } e \geqq 0 \tag{A-5}$$

where $P(-e) = P(e)$, $G = \lambda/2$, and $\lambda^2 = 2/\overline{e^2}$. With this choice of $P(e)$, the solution of (A-2)† is

$$(v/V) = \frac{1 - \exp[(-\sqrt{2}C/3)(e/V)]}{1 - \exp(-\sqrt{2}C/3)} \tag{A-6}$$

Thus, for any given relative volume (i.e., for each value of $C = V/(\overline{e^2})^{1/2}$), (A-6) specifies the compression characteristic required to minimize the quantizing error power.

We are therefore led to study the properties of the family of characteristics of the form

$$(v/V) = \frac{1 - \exp(-me/V)}{1 - \exp(-m)} \quad \text{for} \quad 0 \leqq e \leqq V \tag{A-7}$$

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* An alternate derivation of (A-2) and (A-4), has been given by Panter and Dite,[5] who also acknowledge a prior and different deduction by P. R. Aigrain. Upon reading a preliminary version of the present manuscript, B. McMillan called my attention to S. P. Lloyd's related, but unpublished work, which proved to contain still another derivation. I am grateful to Dr. Lloyd for access to this material.

† In the vocabulary of analytical dynamics, the direct integrability of the Euler equation may be ascribed to the existence of an "ignorable" or "cyclic" coordinate.[29]

TABLE I—COMPARISON OF THE "μ" AND "m" COMPANDOR ENSEMBLES

Property	μ-Ensemble	m-Ensemble
Defining requirement	Approaches uniform precision in the quantization of all pulse samples outside the unavoidable linear region for small samples	Minimizes quantizing error power for a specific volume, if assume an exponential amplitude distribution for that volume
Compression equation, for $0 \leq e \leq V$, where $v(-e) = -v(e)$	$\left(\frac{v}{V}\right) = \frac{\log\left(1 + \frac{\mu e}{V}\right)}{\log(1+\mu)}$	$\left(\frac{v}{V}\right) = \frac{1 - \exp(-me/V)}{1 - \exp(-m)}$
Sample to step size ratio	$\left(\frac{e}{\Delta e}\right) = N/2(1 + V/\mu e) \log(1+\mu)$	$\left(\frac{e}{\Delta e}\right) = (N/2M)(me/V) \exp(-me/V)$ where $M = 1 - \exp(-m)$
	$\left(\frac{e}{\Delta e}\right) \to [N/2 \log(1+\mu)](\mu e/V)$ for $(e/V) \ll \mu^{-1}$	$\left(\frac{e}{\Delta e}\right) \to (N/2M)(me/V)$ for $(e/V) \ll m^{-1}$
	$\left(\frac{e}{\Delta e}\right) \to N/2 \log(1+\mu)$ for $(e/V) \gg \mu^{-1}$	$\left(\frac{e}{\Delta e}\right) = \text{MAX at } (e/V) = m^{-1}$
Ratio of largest to smallest step size = ratio of compression characteristic slopes for zero and overload inputs	$\frac{(\Delta e)_{e=V}}{(\Delta e)_{e=0}} = \frac{(dv/de)_{e=0}}{(dv/de)_{e=V}} = \mu + 1$	$\frac{(\Delta e)_{e=V}}{(\Delta e)_{e=0}} = \frac{(dv/de)_{e=0}}{(dv/de)_{e=V}} = \exp(m)$

Saturated companding improvement, corresponding to linear compression of weakest signals	$C \gg \mu: 20 \log_{10}\left[\frac{\mu}{\log(1+\mu)}\right] db$	$C \gg m: 20 \log_{10}\left[\frac{m}{1-\exp(-m)}\right] db$
Relation between relative signal strength and critical compression parameter which provides greatest reduction of quantizing error for a given volume	$C^2S + \mu_c AC(S-1) - \mu_c^2 = 0$ where $C = V/\sqrt{\overline{e^2}}$ = relative signal strength $S = \left(\frac{1+\mu_c}{\mu_c}\right)\log(1+\mu_c) - 1$ $A = \overline{\vert e\vert}/\sqrt{\overline{e^2}}$ = indep. of volume	$C = 3m_c/\sqrt{2}$
$D^2 = \frac{\text{Error Power}}{\text{Signal Power}}$ when $\bar{e} = 0$	$D^2 = \frac{[\log(1+\mu)]^2}{3N^2}\left[1 + \frac{C^2}{\mu^2} + \frac{2AC}{\mu}\right]$	$D^2 = \frac{\sqrt{2}C^3M^2}{3N^2m^2}\left[\frac{\exp(2m-\sqrt{2}C)-1}{2m-\sqrt{2}C}\right]$

with $v(-e) = -v(e)$ as usual. The "m-characteristics" specified by (A-7) are to be compared with the "μ-characteristics" specified by (8).

From the derivation of (A-6) it is known that optimum companding will be produced when m is given by the critical value,

$$m_c = \sqrt{2}C/3 \tag{A-8}$$

This is the analogue of (36) defining μ_c for the μ-ensemble.

3. Properties of the "m-Ensemble"

We shall now interpret the properties of the m-ensemble of compandors, for which the ensemble improvement limit ($m = m_c$) actually

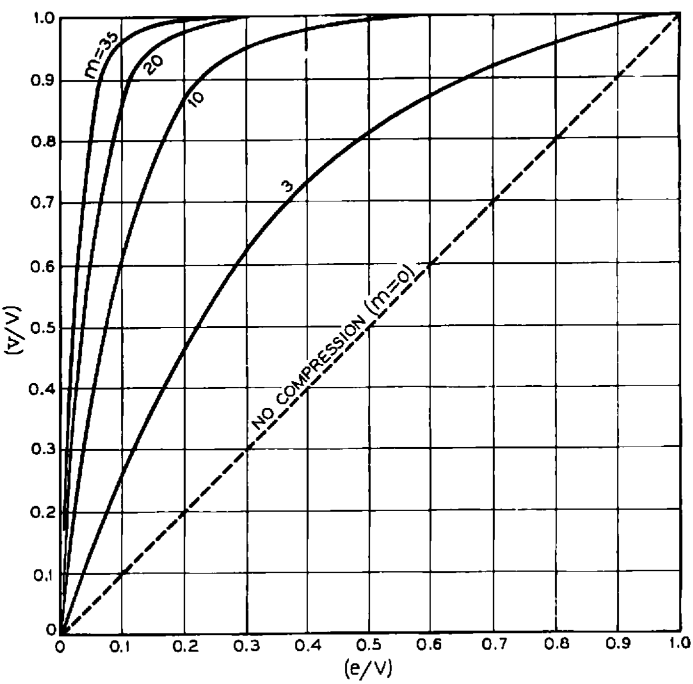

Fig. 21 — Typical m-ensemble compression characteristics determined by equation (A-7). Note the strong emphasis on weak signal amplitudes. These curves may be compared with those for the μ-ensemble in Fig. 3.

minimizes the total quantizing error power, when the probability density is specified by (A-5). Table I summarizes the important properties which may be derived by replacing (8) by (A-7) in the previous detailed analysis of the μ-ensemble.

(a) Compression Characteristics

Compression characteristics, corresponding to various values of m are displayed in Figs. 21 and 22 for direct comparison with the curves in Figs. 3 and 4. The m-characteristics assign very little weight to the larger signal amplitudes in view of the infrequent occurrence of the latter.

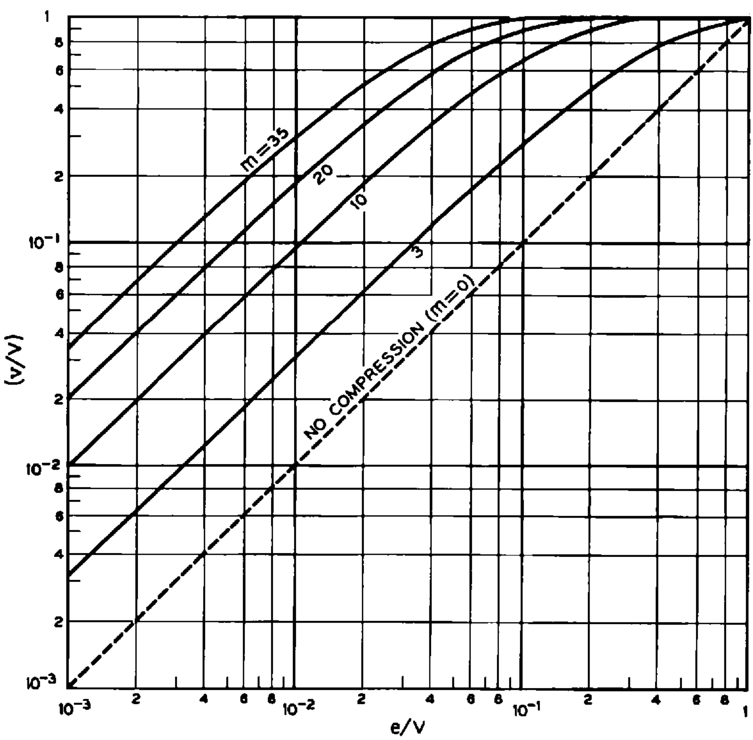

Fig. 22 — Logarithmic replot of compression curves of the type shown in Fig. 21 to indicate detailed behavior for weak samples. These may be compared with the μ-ensemble curves in Fig. 4.

(*b*) *Sample to Step Size Ratio*

Fig. 23, where the sample to step size ratio ($e/\Delta e$) is plotted in the same manner as in Fig. 5, reveals the relative quantizing accuracy accorded various pulse amplitudes.

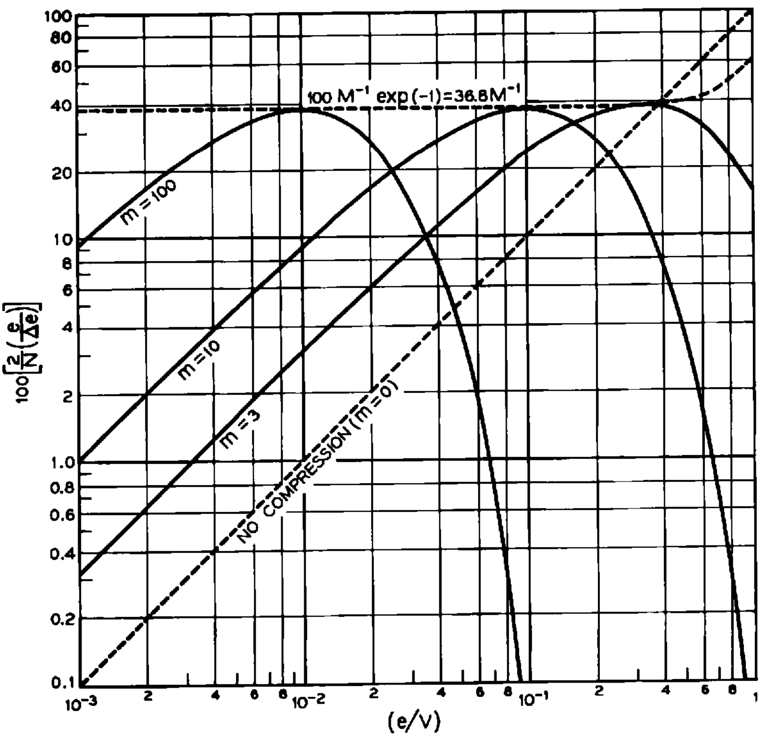

Fig. 23 — Pulse sample to step size ratios as a function of relative sample amplitude, for various compandors in the m-ensemble. The maxima exhibited by these curves occur at $e/V = m^{-1}$; $M = 1 - \exp(-m)$. Compare with Fig. 5.

(c) *Saturated Improvement of Weak Signals*

For signals whose largest samples are confined to the region $(e/V) \ll m^{-1}$, compression is linear, with a saturation improvement noted in Table I and plotted in Fig. 24 for comparison with Fig. 9.

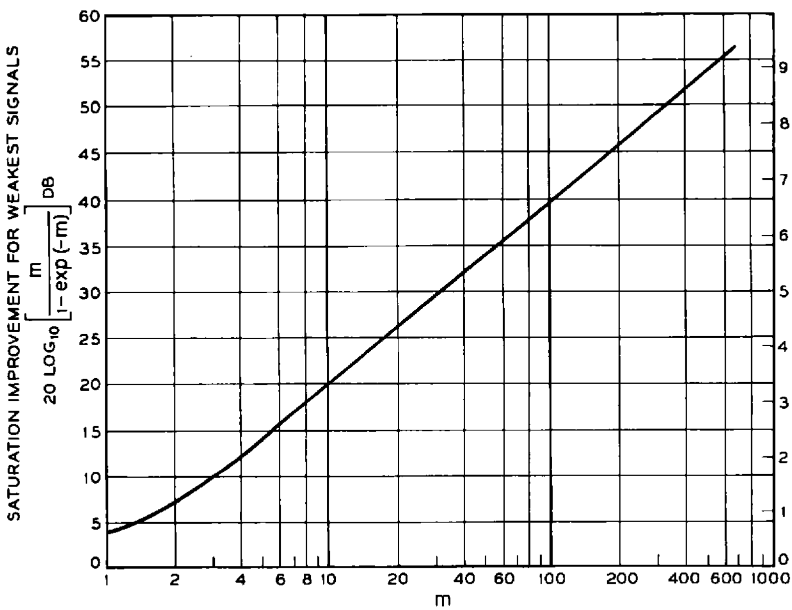

Fig. 24 — Saturated companding improvement for the weakest signals as a function of the degree of "m-type" compression. Given a value of m, the corresponding ordinate represents the reduction of quantizing error power (in db) which results from companding of signals so weak that signal peaks satisfy the relation $(e/V) \ll m^{-1}$. Thus, weaker and weaker signals are required to exploit the added improvement which follows from an increase in m. This curve may be compared with that for the μ-ensemble in Fig. 9.

(d) Variation of Companding Improvement with Volume

Companding improvement curves are shown in Fig. 25 for representative members of the m-ensemble. Each curve is tangent to the ensemble upper limit at the volume for which $m = m_c$. In view of its deduction as the solution of the variational problem, this upper limit actually represents the absolute maximum value of companding improvement for the present choice of $P(e)$.

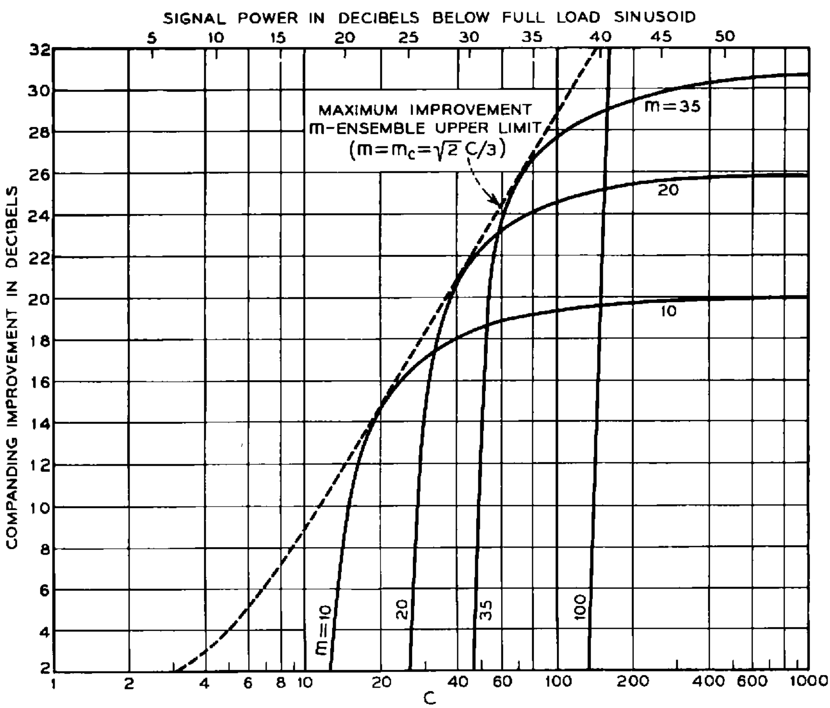

Fig. 25 — Companding improvement curves for representative members of the m-ensemble. These curves are to be compared with those for the μ-ensemble in Fig 8. Note the important difference between the two ensembles for strong signals (small values of C).

(*e*) *Signal to Quantizing Error Power Ratios*

The curves in Fig. 26 are drawn for the representative case of $N = 2^7 = 128$ quantizing steps (7 digit PCM). The corresponding ensemble limit is constant, as might be expected from (A-4), except for strong signals where the effects of peak clipping become noticeable.

In the region where this ensemble limit is constant, departures from the improvement limit resulting from the use of a single value of m for all volumes may be read directly from the ordinates shown at the right in Fig. 26. In comparing these departures from maximum improvement with the analogous μ-ensemble curves in Fig. 14, it must always be recalled that, in view of its role in the solution of the variational problem, the m-ensemble limit represents the actual minimum quantizing error power consistent with the probability density specified by (A-5).

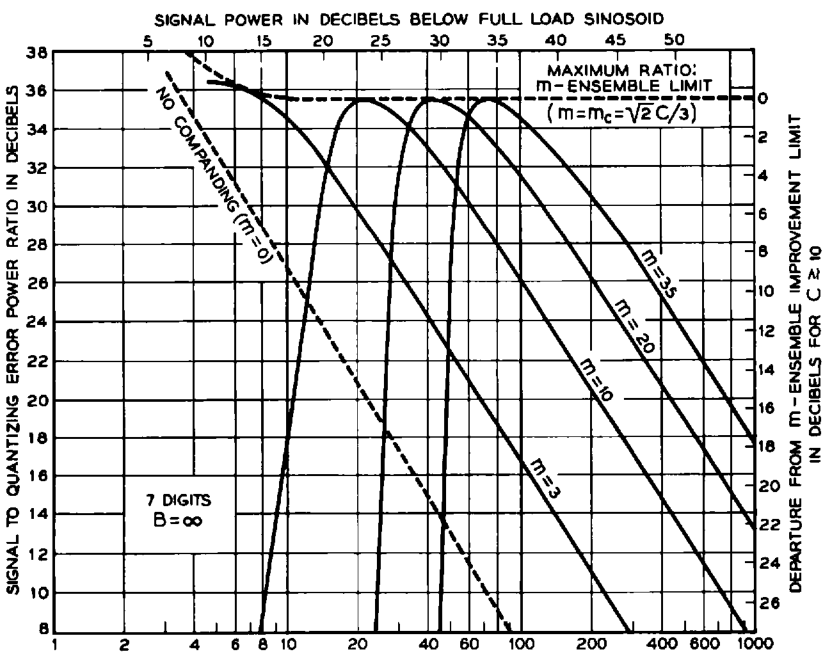

Fig. 26 — Signal to quantizing error power ratios as a function of relative signal power for 7 digits and various m-compandors. The curves may be compared with those for 7 digits in Figs. 15 to 18. The auxiliary ordinates at the right of the present figure apply for $C \gtrsim 10$, where the m-ensemble limit is effectively constant; departures from this limit, resulting from the use of a single value of m for all volumes, may be read directly from this scale, for comparison with Fig. 14. The latter comparison illustrates the narrow volume limitation of the members of the m-ensemble.

(*f*) *Illustrative Application*

Consider the possibility of choosing a member of the m-ensemble for application to the hypothetical PCM system already discussed in connection with the μ-ensemble. It will be recalled (see Figs. 15–18) that we were able to choose degrees of logarithmic compression which would yield signal to quantizing error power ratios in excess of about 20 db for all volumes ($4.5 \lesssim C \lesssim 450$) by using as few as six or seven (depending on the choice of μ) digits per code group. In contrast, Fig. 26 reveals that no value of m will meet this requirement since the curves fall so rapidly on either side of the sharp maxima. In short, the members of the m-ensemble are each too specialized for successful application to such a broad volume range.

Further detailed comparison between the numerical results for the two ensembles seems inappropriate, since it is not at all clear that the inequitable treatment of the various samples in a given signal by members of the m-ensemble (see Fig. 23) permits an adequate description of signal quality solely in terms of quantizing error power. Under these circumstances, subjective effects beyond the scope of the present analysis might assume a dominant role.

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